STATE SPACE IDENTIFICATION OF CIVIL ENGINEERING STRUCTURES FROM OUTPUT MEASUREMENTS

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ABSTRACT

This paper presents the results from a state space system identifcation simulation study of a 5-degrees-of freedom system driven by white noise. The aim of the study is to compare the durability of the fairly new Stochastic Subspace Technique (SST) with more well-known techniques for identification of civil engineering structures. The SST is compared with the stochastic realization estimator Matrix Block Hankel (MBH) and a prediction error method (PEM). The results show that the investigated techniques give good results in terms of estimated modal parameters and mode shapes. Especially, it is found that the new SST technique gives quickly good results compared with the PEM which takes more time with only a limited improvement of the fit on data.

NOMENCLATURE

M	Mass matrix
K	Stiffness matrix
C	Damping matrix
$z(t)$	Displacement vector
$x(t)$	State vector
$y(t)$	Measurement vector
t	Time
F	Continuous-time eigenvector matrix
E	Input matrix
A	State transition matrix
B	Input matrix
C	Observation Matrix
D	Auxillary observation matrix
ζ_j	Damping ratio of the jth mode
f_j	Eigen-frequency of the jth mode
μ	Discrete-time eigenvalue matrix
ψ	Eigenvector
λ	Continuous-time eigenvalue matrix
$w(t)$	Process noise
$v(t)$	Observaton noise
$Y_{i\|2i-1}$	Row space
Z_i	Orthogonal projection of the row space of $Y_{i\|2i-1}$
$H_{jk}(\,)$	Hankel matrix
Q_j	Observability matrix
Γ_k	Covariance matrices

1. INTRODUCTION

System identification is generally the art of mathematical modelling given input-output data from a dynamic system. System identification techniques are often originated from electrical engineering and thereafter adapted to civil engineering. Recently, subspace-based methods for system identification have attracted much attention in electrical engineering, see e.g. Van Overschee et al. [1], De Moor et al. [2]. This interest is due to the ability of providing accurate state-space models for multivariate linear systems directly from measured data. The methods have their origin in classical state-space realization theory as developed in the 1960's, see e.g. Aoki [3]. The main theorem of the subspace theory demonstrates how the Kalman filter states can be obtained from input-output data using linear algebra tools (QR and SVD). Once these states are known, the identification problem becomes a linear least squares problem in the unknown system matrices. The parametrization is easy and convergence is not iterative and guaranteed. The subspace-methods do not rely on solving a highly nonlinear optimization problem and a canonical parametrization as e.q. used with Auto-Regressive-Moving Average Vector (ARMAV) models, see e.g. Andersen et al. [4], Kirkegaard et el. [5]. Comparing to the traditional stochastic realization methods such as e.g. the Matrix Block Hankel (MBH) and the Eigenvalue Realization Algorithm (ERA) the subspace techniques are data driven instead of covariance driven, so that the explicit formation of the covariance matrix is avoided. If the external input is unknown a Stochastic Subspace Technique (SST) can be used to determine the system matrices, Van Overschee et al. [1]. This technique could be useful for the determination of the modal parameters of civil engineering structures which often are excited by unmeasured ambient loading. A first attempt to use SST in civil engineering was presented in Peeters et al. [6] where promissing results were obtained. The aim of the present paper is to evaluate the durability of this new SST for identification of civil engineering structures. The SST is compared with the more well-known stochastic realization estimators Matrix Block Hankel (MBH) , see e.g. Hoen [7]. Further, these state-space identification techniques are compared with a prediction error method (PEM) The comparisons in the paper are based on simulated data from a 5-degrees-of freedom system driven by white noise.

2. STATE SPACE MODELLING OF STRUCTURAL SYSTEMS

Consider a multivariate continuous-time civil engineering second order structural system with *ny* degrees of freedom. The system is assumed to be time-invariant and described by a positive definite diagonal mass matrix $\boldsymbol{M}$, a symmetric semi-definite viscous damping matrix $\boldsymbol{C}$, and a symmetric positive definite stiffness matrix $\boldsymbol{K}$. For the present purpose the system is excited by a, for the present purpose, deterministic load vector $\boldsymbol{u}(t)$ through a selection matrix $\boldsymbol{S}$. Denoting the zero-mean response vector by $\boldsymbol{z}(t)$, the differential equation of this system is

$$\ddot{z}(t) + \boldsymbol{M}^{-1}\boldsymbol{C}\dot{z}(t) + \boldsymbol{M}^{-1}\boldsymbol{K}z(t) = \boldsymbol{M}^{-1}\boldsymbol{S}\boldsymbol{u}(t) \qquad (1)$$

Stacking the identity $\dot{z}(t) = \dot{z}(t)$ and (1) on top of each other yields the state equation

$$\begin{bmatrix} \dot{z}(t) \\ \ddot{z}(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} & I \\ -\boldsymbol{M}^{-1}\boldsymbol{K} & -\boldsymbol{M}^{-1}\boldsymbol{C} \end{bmatrix} \begin{bmatrix} z(t) \\ \dot{z}(t) \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{M}^{-1}\boldsymbol{S} \end{bmatrix} \boldsymbol{u}(t) \qquad (2)$$

Compactly this equation can be written as

$$\dot{x}(t) = \boldsymbol{F}x(t) + \boldsymbol{E}\boldsymbol{u}(t), \quad x(0) = x_0 \qquad (3)$$

The vector $\boldsymbol{x}(t)$ is the state vector, which in the present case consists of nodal displacements and velocities. The matrix $\boldsymbol{F}$ is the state matrix, whereas the matrix $\boldsymbol{E}$ is referred to as the input matrix. The state matrix $\boldsymbol{F}$ comprises all information about the system and provides as such a full description of the system.

The system response of the state space system can be observed through an observation vector $\boldsymbol{y}(t)$. The response depends on the type of measurements and the process to be measured. In the present case the response will typically be either displacements, velocities, or accelerations. In the general case the response $\boldsymbol{y}(t)$ will be a linear combination of these three cases. Let ξ_d, ξ_v and ξ_a be coefficients that weight the contributions of the displacements, velocities and accelerations, respectively. The response is then obtained from the following equation

$$\begin{aligned} y(t) &= \xi_d z(t) + \xi_v \dot{z}(t) + \xi_a \ddot{z}(t) \\ &= \big[[I \;\; \boldsymbol{0}]\xi_d + [\boldsymbol{0} \;\; I](\xi_v + \xi_a \boldsymbol{F}) \big] x(t) + [\boldsymbol{0} \;\; I]\xi_a \boldsymbol{E}\boldsymbol{u}(t) \qquad (4) \\ &= \boldsymbol{C}x(t) + \boldsymbol{D}\boldsymbol{u}(t) \end{aligned}$$

It is seen that the general system output includes a linear combination of the state variables and a direct term $\boldsymbol{D}\boldsymbol{u}(t)$. Equation (4) is named the observation equation. The combination of the state space and the observation equations fully describes the input and output behaviour of the continuous-time structural system and is as such named the state space system.

2.1 Sampled deterministic state space system

The discrete-time sampled version of the state space system can be obtained by applying a zero-order hold approximation. The discrete-time state space system is then obtained as

$$\begin{aligned} x(t+1) &= \boldsymbol{A}x(t) + \boldsymbol{B}\boldsymbol{u}(t), \quad x(0) = x_0 \\ y(t) &= \boldsymbol{C}x(t) + \boldsymbol{D}\boldsymbol{u}(t) \end{aligned} \qquad (5)$$

where the index t now signifies discrete time instance and $\boldsymbol{x}(t)$ signifies the discrete-time zero-mean state vector. The state matrix $\boldsymbol{A}$ and the input matrix $\boldsymbol{B}$ are given by

$$\boldsymbol{A} = e^{\boldsymbol{F}T}, \quad \boldsymbol{B} = \boldsymbol{F}^{-1}(\boldsymbol{A} - \boldsymbol{I})\boldsymbol{E} \qquad (6)$$

with T being the sampling interval, whereas the discrete-time observation matrix $\boldsymbol{C}$ and direct term $\boldsymbol{D}$ is unaffected by the sampling.

The system in (5), however, does not account for disturbance that affects the structural parameters describing the system. This disturbance affects the states and can e.g. be environmental variations, such as the temperature causing changes of the stiffness of the structure, or other modelling inaccuracies. This disturbance is termed process noise and is denoted $\boldsymbol{w}(t)$. Further, when a system is sampled there will most certainly be introduced some observation noise $\boldsymbol{v}(t)$, due to e.g. limited measurement accuracy, or sensor inaccuracies

2.2 Stochastic discrete-time state space system

By treating the noise terms $\boldsymbol{w}(t)$ and $\boldsymbol{v}(t)$ as stochastic processes, and by adding these to the state space equation and the observation equation, respectively, the stochastic state space system is defined as

$$\begin{aligned} x(t+1) &= \boldsymbol{A}x(t) + \boldsymbol{B}\boldsymbol{u}(t) + w(t), \quad x(0) = x_0 \\ y(t) &= \boldsymbol{C}x(t) + \boldsymbol{D}\boldsymbol{u}(t) + v(t) \end{aligned} \qquad (7)$$

The difference between this state space system and (5) is that it includes the two stochastic terms, which are the reason for its name. Both $\boldsymbol{w}(t)$ and $\boldsymbol{v}(t)$ are assumed to be weakly stationary and independent identically distributed random variables with zero-mean, which implies

$$E\begin{bmatrix} w(t) \\ v(t) \end{bmatrix} = \boldsymbol{0}, \quad E\left[\begin{bmatrix} w(t) \\ v(t) \end{bmatrix} \begin{bmatrix} w^T(j) & v^T(j) \end{bmatrix}\right] = \begin{bmatrix} \boldsymbol{Q} & \boldsymbol{S} \\ \boldsymbol{S}^T & \boldsymbol{R} \end{bmatrix} \delta(t-j) \qquad (8)$$

where δ is the Kronecker delta function. $\boldsymbol{Q}$ and $\boldsymbol{R}$ are assumed to be positive definite and semi-definite respectively. The measurement noise $\boldsymbol{v}(t)$ will in the general case be correlated with the process noise $\boldsymbol{w}(t)$, see Ljung [11] and Hannan et al. [10].

In some cases only the output $y(t)$ is known. This is the typical case in system identification of ambient excited civil engineering structures. This implies by omitting the input term that the state space system looks as follows

$$x(t+1) = Ax(t) + w(t)\,, \quad y(t) = Cx(t) + v(t) \qquad (9)$$

3. IDENTIFICATION OF STATE-SPACE MODELS

This section presents three state space system identification techniques which can be used to identify a system only from output measurements. The results of the estimation techniques are the state space matrices {*A*, *C*}.

3.1 Stochastic Subspace Technique

Subspace algorithms for identification of linear dynamic systems have recently been considered in a number of papers, see Van Overshee et al. [1] and De Moor et al. [2]. The main theorem of the subspace theory demonstrates how the Kalman filter states can be obtained from input-output data using linear algebra tools (QR and SVD). Once these states are known, the identification problem becomes a linear least-squares problem in the unknown matrix pair {*A*, *C*}. If the external input is unknown, a stochastic subspace technique (SST) is used to determine the system matrices. Compared to the stochastic realization methods in the following section, the SST is data driven instead of covariance driven, so that the explicit formation of the covariance matrix is avoided. In the following the SST is briefly described based on Van Overschee et al. [1] and De Moor. Et al. [2].

In order to use the SST it is assumed that the system is observable. The SST relies on output block Hankel matrices of the form

$$Y_{0|i-1} = \begin{bmatrix} y(0) & y(1) & . & . & y(j-1) \\ y(1) & y(2) & . & . & y(j) \\ & & & . & . \\ y(i-1) & y(i) & . & . & y(i+j-2) \end{bmatrix} \qquad (10)$$

where the first subscript denotes the time index of the upper left element, while the second subscript is the time index of the bottom left element. For all output block Hankel matrices, the number of columns will be j, and for all theoretical derivations it is assumed that $j \to \infty$.

An orthogonal projection Z_i of the row space of $Y_{i|2i-1}$ (the future) onto the row space of $Y_{0|i-1}$ (the past) is introduced

$$Z_i \equiv Y_{i|2i-1} / Y_{0|i-1} \qquad (11)$$

By singular value decomposition of this projection it can be proved that

$$Z_i = Q_i \hat{X}_i \qquad (12)$$

which is the product of the extended observability matrix $\boldsymbol{Q}_i$

$$O_j = \begin{bmatrix} C^T & (CA)^T & (CA^2)^T & & (CA^{j-1})^T \end{bmatrix}^T \qquad (13)$$

and $\hat{X}_i \equiv [\hat{x}(t)\ \hat{x}(t+1)\ \ldots\ \hat{x}(t+j-1))]$, which is the Kalman state sequence. Further, it can be proved that another projection Z_{i+1} is defined as

$$Z_{i+1} \equiv Y_{i+1|2i-1} / Y_{0|i} \qquad (14)$$

implying that

$$Z_i = Q_{i-1} \hat{X}_{i+1} \qquad (15)$$

From (13) and (15) the Kalman states can be obtained from output data using singular value decomposition techniques which mean that the matrix pair {*A*, *C*} can be estimated from the following set of linear equations

$$\begin{bmatrix} \hat{X}_{i+1} \\ Y_{i|i} \end{bmatrix} = \begin{bmatrix} A \\ C \end{bmatrix} \hat{X}_i + \begin{bmatrix} \rho_w \\ \rho_v \end{bmatrix} \qquad (16)$$

where the last term consists of residual matrices. A numerically efficient way to solve this set of linear equations is described in Van Overschee et al. [1].

3.2 Matrix Block Hankel Stochastic Realization Estimator

The Matrix Block Hankel stochastic realization estimator is a covariance driven estimation technique. The present algorithm is based on Aoki [3], whereas the name of the technique is due to Hoen [8].

The estimation of the matrices {*A*, *C*} is based on a decomposition of a block Hankel matrix of the form

$$H_{jk}(p) = \begin{bmatrix} \Gamma_p & \Gamma_{p+1} & . & \Gamma_{p+k-1} \\ \Gamma_{p+1} & \Gamma_{p+2} & . \; . & \Gamma_{p+k} \\ . & . & . \; . & . \\ \Gamma_{p+j-1} & \Gamma_{p+j} & . & \Gamma_{p+j+k-2} \end{bmatrix} \qquad (17)$$

which for p=1 is the product of the observability matrix $\boldsymbol{Q}_j$ and another matrix called Ω_k, defined as

$$\Omega_k = \begin{bmatrix} M & AM & A^2M & . \; . & A^{k-1}M \end{bmatrix} \qquad (18)$$

First the singular value decomposition $\boldsymbol{USV}^T$ of the Hankel matrix $\boldsymbol{H}_{jk}(1)$ is calculated. In this matrix the theoretical covariance matrices Γ_k of the response have been replaced by sampled covariance matrices. By a proper choice of coordinate system an internal balanced estimate of the triple is then obtained

$$\hat{A} = S^{-1/2} U^T H_{jk}(2) V S^{-1/2}$$
$$\hat{C} = H_{1k}(1) V S^{-1/2} \qquad (19)$$

The constants j and k will in general be equal to the state space dimension divided by the number of observed outputs. For minimal systems the estimates are unique, because the Gramians $O_j^T O_j$ and $\Omega_k \Omega_k^T$ are non-singular.

3.3 Prediction Error Methods

The Prediction Error Method (PEM) contains the most known and used methods for system identification and has been developed and analysed extensively during the last three decades, see e.g. Ljung [9] and Andersen et al. [7]. In order to use PEM for system identification, a parametrization of the state space matrices in (9) is made to obtain the predictor of the state space system output $\hat{y}(t)$. The parameters of the model are calculated by minimizing a quadratic loss function V of the prediction errors $\varepsilon(t) = y(t) - \hat{y}(t)$, giving

$$V = \frac{1}{2}\frac{1}{N}\sum_{k=1}^{N} \varepsilon^T(t_k) W^{-1} \varepsilon(t_k) \qquad (20)$$

where W is a weight matrix, and N is the number of output measurements. The minimization of (19) is in general nonlinear in the parameters and has to be solved by an iterative method, e.g. Gauss-Newton algorithm. This optimization depends highly upon the initial guess of the parameter vector, see e.g. Ljung [9].

4. EXAMPLE

In the following example the performance of the SST is compared with the performance of MBH and PEM:The reference system is a 5-dof linear system excited by Gaussian white noise. The reference system has been converted, using a sampling interval of 0.015 seconds, from a continuous-time description to a covariance equivalent discrete time ARMAV model, see Andersen et al. [4]. The eigenfrequcies f_i and damping ratios$_i$ ζ of the simulated system are

$$f_i = \{2.1405,\ 7.5897,\ 13.2922,\ 17.9697,\ 21.0034\}$$
$$\zeta_i = \{0.0124,\ 0.0134,\ 0.0261,\ 0.0271,\ 0.0331\}$$

The modal parameters can be determined from the following eigenvalue problem

$$(I\mu_j - A)\Psi_j\,, \quad \Phi_j = C\Psi_j\,, \quad j = 1, 2, \dots, n \qquad (22)$$

where Ψ_j is the jth eigenvector of the system, and μ is the corresponding eigenvalue. The $ny \times 1$ vector Φ_j is the observed part of the eigenvector, and referred to as the scaled mode shape. For a stable underdamped system all structural modes are represented by complex conjugated pairs of eigenvalues and corresponding mode shapes. The complex conjugated pair of eigenvalues $\{\mu_j, \mu^*_{j+1}\}$ can be equivalently expressed in terms of eigenfrequencies f_i and damping ratios ζ_i by converting each of them to continuous-time eigenvalues $\lambda_j = \log(\mu_j)/T$, resulting in

$$\{\lambda_j, \lambda^*_{j+1}\} = -2\pi f_j \zeta_j \pm i2\pi f_j \sqrt{1-\zeta_j^2}$$
$$f_j = \frac{|\lambda_j|}{2\pi}\,, \quad \frac{-Re(\lambda_j)}{2\pi f_j} \qquad (23)$$

The estimated mean values and standard deviation of the eigenfrequencies and damping ratios are shown in rows 1 to 15 and 16 to 30, respectively, in tables 1, 2, 3 and 4 from 25 simulation runs with N ={1250, 2500, 5000, 10000} samples, respectively. Gaussian white noise has been added to the output of the system amounting to σ_0 = {1, 5, 10} % of the mean of the standard deviations of the simulated output. Further, the mean values and standard deviation of the trace of the MAC-matrice are shown in the tables in rows 31 to 33. The Modal Assurance Criteria (MAC) which is a commonly used method for assessing the degree of correlation between any two eigenvectors is given by

$$MAC_{ij} = \frac{[\phi_i^H \phi_j]^2}{[\phi_i^H \phi_i]^2 [\phi_j^H \phi_j]^2} \qquad (24)$$

where H is the complex transposed

The identification is performed using a state space order equal to 10 for both SST and PEM. However, the optimal state space dimension for the MBH estimator is 30. This dimension results in the estimation of some spuerious modes in the MBH estimation. Physical modes were selected using stabilization diagrams.

The tables 1-4 show that both the eigenfrequencies have been identified very well. Especially, the SST and the PEM estimates are very close to the modal parameters of the reference system. The MBH method is seen to give poor estimates of the damping ratios and the mode shapes compared with the two other techniques. Further, it is very interesting that the SST gives estimates of both modal parameters and modes shapes which are of same quality as the estimates obtained by the PEM. The SST is approximately ten times faster then the PEM.

5. CONCLUSION

This paper presents the results from a state space system identifcation simulation study of a 5-degrees-of freedom system driven by white noise. The aim of the study was to compare the durability of the fairly new Stochastic Subspace Technique (SST) with more well-known techniques for identification of civil engineering structures. The results have shown that the investigated techniques give useable results in terms of estimated modal parameters and mode shapes. Especially, the PEM and the SST algorithms reveals a high degree of agreement with the modal parameters and mode shapes of the reference system. Further, it is found that the new SST technique gives quickly good results compared with the PEM which takes more time with only a limited improvement of the fit on data.

6. ACKNOWLEDGEMENTS

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	σ_n	SST	MBH	PEM
1	1	2.1450 (0.0161)	2.1440 (0.0531)	2.1451 (0.0163)
1	5	2.1454 (0.0159)	2.1336 (0.0593)	2.1459 (0.0158)
1	10	2.1457 (0.0155)	2.1597 (0.0914)	2.1476 (0.0164)
2	1	7.5824 (0.0248)	7.5932 (0.0495)	7.5847 (0.0261)
2	5	7.5816 (0.0266)	7.5639 (0.0648)	7.5843 (0.0267)
2	10	7.5827 (0.0298)	7.5906 (0.0726)	7.5847 (0.0298)
3	1	13.2897 (0.0602)	13.2722 (0.1490)	13.2869 (0.0643)
3	5	13.2824 (0.0592)	13.2919 (0.1616)	13.2764 (0.0687)
3	10	13.2723 (0.1001)	13.1830 (0.3270)	13.2687 (0.0928)
4	1	17.9603 (0.0580)	17.9786 (0.1798)	17.9647 (0.0606)
4	5	17.9692 (0.0892)	17.9804 (0.2467)	17.9620 (0.0739)
4	10	18.1750 (0.5269)	17.9212 (0.5188)	17.9847 (0.1714)
5	1	21.0065 (0.1325)	20.9651 (0.3051)	21.0273 (0.0952)
5	5	21.0654 (0.7746)	21.1264 (0.3549)	21.0277 (0.3443)
5	10	19.8127 (1.6803)	21.0209 (0.8074)	18.2262 (0.8541)
1	1	0.0118 (0.0060)	0.0199 (0.0152)	0.0117 (0.0057)
1	5	0.0120 (0.0058)	0.0216 (0.0156)	0.0118 (0.0057)
1	10	0.0121 (0.0059)	0.0383 (0.0577)	0.0124 (0.0064)
2	1	0.0149 (0.0050)	0.0525 (0.1627)	0.0150 (0.0047)
2	5	0.0151 (0.0047)	0.0611 (0.1993)	0.0151 (0.0045)
2	10	0.0153 (0.0051)	0.0588 (0.1964)	0.0151 (0.0050)
3	1	0.0268 (0.0037)	0.0476 (0.0532)	0.0270 (0.0035)
3	5	0.0276 (0.0049)	0.0316 (0.0225)	0.0271 (0.0053)
3	10	0.0289 (0.0076)	0.0606 (0.1014)	0.0271 (0.0073)
4	1	0.0272 (0.0047)	0.0459 (0.0751)	0.0269 (0.0050)
4	5	0.0302 (0.0090)	0.0284 (0.0131)	0.0283 (0.0063)
4	10	0.0386 (0.0225)	0.0322 (0.0217)	0.0312 (0.0189)
5	1	0.0372 (0.0057)	0.1149 (0.2690)	0.0358 (0.0055)
5	5	0.0446 (0.0217)	0.0709 (0.1947)	0.0569 (0.0959)
5	10	0.0454 (0.0260)	0.0623 (0.0733)	0.0319 (0.0189)
M	1	4.9944 (0.0026)	4.4573 (0.4741)	4.9955 (0.0028)
A	5	4.9203 (0.1765)	4.5485 (0.2469)	4.9667 (0.0341)
C	10	4.4788 (0.3775)	4.1331 (0.4368)	4.1574 (0.2234)

Table 1: Mean values and standard deviation of f_i, ζ and tr (MAC) for N=1250

	σ_n	SST	MBH	PEM
1	1	2.1424 (0.0083)	2.1541 (0.0782)	2.1425 (0.0078)
1	5	2.1426 (0.0083)	2.1489 (0.0418)	2.1424 (0.0080)
1	10	2.1427 (0.0083)	2.1360 (0.0479)	2.1427 (0.0087)
2	1	7.5972 (0.0218)	7.5824 (0.0615)	7.5968 (0.0195)
2	5	7.5964 (0.0224)	7.5969 (0.0470)	7.5974 (0.0209)
2	10	7.5972 (0.0247)	7.6066 (0.0473)	7.5974 (0.0240)
3	1	13.2880 (0.0331)	13.3208 (0.0840)	13.2891 (0.0354)
3	5	13.2955 (0.0435)	13.3204 (0.1339)	13.2962 (0.0498)
3	10	13.3081 (0.0578)	13.2839 (0.1754)	13.3014 (0.0631)
4	1	17.9777 (0.0408)	17.9871 (0.0786)	17.9790 (0.0422)
4	5	17.9740 (0.0464)	17.9102 (0.1549)	17.9770 (0.0476)
4	10	18.0020 (0.1449)	17.9055 (0.2736)	17.9673 (0.0948)
5	1	20.9950 (0.0647)	21.0285 (0.2053)	21.0089 (0.0662)
5	5	21.0084 (0.1176)	21.1290 (0.3681)	21.0091 (0.1080)
5	10	20.5411 (1.4402)	20.9262 (0.6846)	19.5217 (1.4602)
1	1	0.0114 (0.0033)	0.0238 (0.0284)	0.0113 (0.0032)
1	5	0.0115 (0.0033)	0.0190 (0.0135)	0.0116 (0.0035)
1	10	0.0114 (0.0033)	0.0154 (0.0115)	0.0117 (0.0035)
2	1	0.0142 (0.0027)	0.0287 (0.0496)	0.0142 (0.0028)
2	5	0.0141 (0.0028)	0.0173 (0.0202)	0.0141 (0.0028)
2	10	0.0138 (0.0030)	0.0405 (0.1195)	0.0138 (0.0031)
3	1	0.0274 (0.0037)	0.0434 (0.0782)	0.0274 (0.0035)
3	5	0.0279 (0.0042)	0.0566 (0.1429)	0.0275 (0.0040)
3	10	0.0293 (0.0053)	0.0599 (0.1052)	0.0423 (0.0684)
4	1	0.0271 (0.0030)	0.0262 (0.0047)	0.0272 (0.0030)
4	5	0.0272 (0.0028)	0.0297 (0.0100)	0.0267 (0.0026)
4	10	0.0289 (0.0066)	0.0734 (0.0808)	0.0269 (0.0050)
5	1	0.0337 (0.0038)	0.0399 (0.0234)	0.0332 (0.0037)
5	5	0.0348 (0.0073)	0.0990 (0.2107)	0.0326 (0.0053)
5	10	0.0627 (0.0564)	0.0580 (0.0796)	0.0670 (0.1995)
M	1	4.9977 (0.0012)	4.7117 (0.3002)	4.9978 (0.0011)
A	5	4.9847 (0.0105)	4.5821 (0.3939)	4.9870 (0.0076)
C	10	4.7308 (0.3231)	4.2011 (0.4395)	4.5021 (0.4956)

Table 2: Mean values and standard deviation of f_i, ζ and tr (MAC) for N=2500

	σ_0	SST	MBH	PEM
1	1	2.1413 (0.0061)	2.1371 (0.0248)	2.1413 (0.0060)
1	5	2.1416 (0.0059)	2.1470 (0.0435)	2.1418 (0.0061)
1	10	2.1417 (0.0059)	2.1496 (0.0168)	2.1419 (0.0060)
2	1	7.5927 (0.0119)	7.5902 (0.0283)	7.5937 (0.0124)
2	5	7.5924 (0.0115)	7.5973 (0.0410)	7.5930 (0.0123)
2	10	7.5926 (0.0125)	7.6063 (0.0450)	7.5929 (0.0130)
3	1	13.2992 (0.0245)	13.3116 (0.0523)	13.2980 (0.0248)
3	5	13.3011 (0.0273)	13.2962 (0.0761)	13.2993 (0.0257)
3	10	13.2989 (0.0415)	13.4078 (0.1827)	13.2973 (0.0400)
4	1	17.9640 (0.0343)	17.9538 (0.0835)	17.9642 (0.0367)
4	5	17.9583 (0.0385)	17.9463 (0.1025)	17.9584 (0.0418)
4	10	17.9548 (0.0561)	18.0242 (0.3203)	17.9587 (0.0639)
5	1	20.9959 (0.0479)	21.0032 (0.1282)	21.0006 (0.0540)
5	5	21.0099 (0.0876)	20.9784 (0.2528)	21.0080 (0.0891)
5	10	21.0991 (0.3089)	20.9010 (0.5524)	20.9906 (0.1583)
1	1	0.0118 (0.0029)	0.0565 (0.1970)	0.0116 (0.0026)
1	5	0.0119 (0.0030)	0.0971 (0.2727)	0.0118 (0.0029)
1	10	0.0119 (0.0030)	0.0564 (0.1968)	0.0120 (0.0031)
2	1	0.0130 (0.0017)	0.0129 (0.0034)	0.0130 (0.0017)
2	5	0.0129 (0.0017)	0.0153 (0.0127)	0.0129 (0.0017)
2	10	0.0128 (0.0017)	0.0190 (0.0354)	0.0128 (0.0018)
3	1	0.0253 (0.0017)	0.0432 (0.0691)	0.0253 (0.0018)
3	5	0.0256 (0.0021)	0.0277 (0.0111)	0.0256 (0.0022)
3	10	0.0261 (0.0026)	0.0510 (0.1053)	0.0257 (0.0027)
4	1	0.0270 (0.0015)	0.0279 (0.0041)	0.0269 (0.0016)
4	5	0.0267 (0.0027)	0.0682 (0.1943)	0.0268 (0.0027)
4	10	0.0270 (0.0049)	0.0316 (0.0285)	0.0263 (0.0046)
5	1	0.0324 (0.0018)	0.0416 (0.0473)	0.0321 (0.0018)
5	5	0.0325 (0.0029)	0.0338 (0.0209)	0.0316 (0.0029)
5	10	0.0364 (0.0091)	0.0641 (0.0752)	0.0301 (0.0055)
M	1	4.9992 (0.0002)	4.8335 (0.2695)	4.9992 (0.0003)
A	5	4.9945 (0.0016)	4.7606 (0.3030)	4.9945 (0.0016)
C	10	4.9658 (0.0146)	4.4195 (0.4256)	4.9679 (0.0138)

Table 3: Mean values and standard deviation of f_i, ζ and *tr* (MAC) for N=5000.

	σ_0	SST	MBH	PEM
1	1	2.1405 (0.0052)	2.1393 (0.0141)	2.1405 (0.0055)
1	5	2.1409 (0.0050)	2.1287 (0.0232)	2.1410 (0.0051)
1	10	2.1409 (0.0050)	2.1417 (0.0251)	2.1411 (0.0053)
2	1	7.5899 (0.0106)	7.5909 (0.0140)	7.5898 (0.0109)
2	5	7.5879 (0.0087)	7.5845 (0.0236)	7.5876 (0.0086)
2	10	7.5880 (0.0093)	7.5700 (0.0531)	7.5878 (0.0087)
3	1	13.2951 (0.0160)	13.2897 (0.0368)	13.2956 (0.0173)
3	5	13.2914 (0.0245)	13.2887 (0.0704)	13.2909 (0.0243)
3	10	13.2899 (0.0289)	13.2939 (0.0759)	13.2887 (0.0280)
4	1	17.9650 (0.0203)	18.0017 (0.1033)	17.9657 (0.0208)
4	5	17.9704 (0.0294)	17.9580 (0.1399)	17.9696 (0.0298)
4	10	17.9707 (0.0514)	18.0207 (0.2040)	17.9676 (0.0524)
5	1	21.0151 (0.0336)	20.9967 (0.1456)	21.0164 (0.0302)
5	5	21.0246 (0.0446)	21.0665 (0.1306)	21.0242 (0.0461)
5	10	21.0310 (0.1206)	20.9377 (0.4746)	21.0231 (0.0975)
1	1	0.0121 (0.0028)	0.0104 (0.0044)	0.0120 (0.0028)
1	5	0.0126 (0.0027)	0.0543 (0.1827)	0.0126 (0.0026)
1	10	0.0126 (0.0027)	0.0274 (0.0577)	0.0126 (0.0026)
2	1	0.0135 (0.0015)	0.0531 (0.1973)	0.0135 (0.0015)
2	5	0.0141 (0.0013)	0.0134 (0.0038)	0.0140 (0.0013)
2	10	0.0141 (0.0014)	0.0181 (0.0177)	0.0140 (0.0014)
3	1	0.0256 (0.0014)	0.0254 (0.0031)	0.0257 (0.0014)
3	5	0.0262 (0.0015)	0.0256 (0.0089)	0.0262 (0.0013)
3	10	0.0264 (0.0020)	0.0482 (0.0754)	0.0263 (0.0017)
4	1	0.0272 (0.0015)	0.0262 (0.0046)	0.0272 (0.0016)
4	5	0.0267 (0.0014)	0.0294 (0.0103)	0.0267 (0.0015)
4	10	0.0264 (0.0017)	0.0323 (0.0231)	0.0263 (0.0019)
5	1	0.0330 (0.0017)	0.0335 (0.0148)	0.0329 (0.0018)
5	5	0.0325 (0.0024)	0.0305 (0.0235)	0.0326 (0.0026)
5	10	0.0329 (0.0055)	0.0431 (0.0483)	0.0325 (0.0045)
M	1	4.9995 (0.0002)	4.9064 (0.1618)	4.9995 (0.0002)
A	5	4.9975 (0.0011)	4.8211 (0.2298)	4.9975 (0.0011)
C	10	4.9853 (0.0071)	4.6050 (0.4058)	4.9855 (0.0066)

Table 4: Mean values and standard deviation of f_i, ζ and *tr* (MAC) for N=10000