ARMA Models in Modal Space

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ABSTRACT

In this papers a new approach for estimation of ARMA models is developed. Based on an analytical transformation between the modal parameters and the ARMA parameters, it is explained how the optimization can be performed in the modal domain. It is explained how the practical estimation problems can be significantly reduced by optimizing on a reduced set of parameters in the modal domain.

NOMENCLATURE

Δt	sampling time step
$\mathbf{y}_t$	response vector
$\mathbf{A}_n, \mathbf{B}_n$	ARMA coefficient matrices
$\mathbf{e}_t$	prediction error
$\mathbf{A}, \mathbf{B}, \mathbf{C}$	state space matrices
$\mathbf{x}_t$	state vector
$\mathbf{I}$	unity matrix
q	shift operator
$\mathbf{H}(.)$	transfer function
$\mathbf{V}$	eigenvector matrix
μ_i, λ_i	poles
$\mathbf{\Phi}, \mathbf{\Psi}$	mode shape matrices
$\phi_\mathbf{i}, \psi_\mathbf{i}$	mode shape vectors
$\mathbf{R}_i$	residue matrix
f, ζ	natural frequency, damping

INTRODUCTION

In prediction error methods (PEM), Ljung [1987], the measured response is modeled by fitting directly to the time series by minimizing the error, i.e. the difference between the measured and the modeled time series. The vector ARMA structure is the simplest possible covariance equivalent model of linear structural systems formulated in discrete time, Andersen et al. [1996], and thus, it is an obvious choice in cases where an optimal estimation of modal parameters is needed. However, since the prediction error can only be minimized using non-linear optimization, and since practical applications often involve many response channels and many modes, the large set of parameters needed to be estimated causes severe problems in calculation time, computer memory management and convergence.

In this paper, a simple way of reducing the number of parameters is considered by performing the optimization on a subset of the model parameters in modal domain. The modal set is then transformed to ARMA domain where the prediction errors are obtained and minimized. First, however, the problems of calibration of ARMA models is briefly explained.

Assume p channels of measurements simultaneously sampled with the time step Δt and ordered in the vector $\mathbf{y}_t$. The subscript t denotes discrete time, i.e. real time is obtained by multiplying t by the time step Δt.

A corresponding vector ARMA model with np poles is given by, Andersen [1997]

$$\mathbf{y}_t + \mathbf{A}_1\mathbf{y}_{t-1} + \ldots + \mathbf{A}_n\mathbf{y}_{t-n} = \mathbf{e}_t + \mathbf{B}_1\mathbf{e}_{t-1} + \ldots + \mathbf{B}_n\mathbf{e}_{t-n} \quad (1)$$

where all $\mathbf{A}_i, \mathbf{B}_i$ are real $p \times p$ matrices, and $\mathbf{e}_t$ is a vector of white noise sequences driving system to response with the sequence $\mathbf{y}_t$. Using PEM estimation, the ARMA model is estimated by predicting the response $\hat{\mathbf{y}}_t$ from linear regression on the past values of the measurements $\mathbf{y}_t$ and noise $\mathbf{e}_t$, i.e. the estimator is, Andersen [1997]

$$\hat{\mathbf{y}}_t = -\mathbf{A}_1\mathbf{y}_{t-1} - \ldots - \mathbf{A}_n\mathbf{y}_{t-n} + \mathbf{B}_1\mathbf{e}_{t-1} + \ldots + \mathbf{B}_n\mathbf{e}_{t-n} \quad (2)$$

Now, assuming, that the measurements can be modeled as given by Equation (1), the differences between the predictions and the measurements are given by the noise sequence $\mathbf{e}_t$, and thus, using least squares estimation, the model is calibrated by minimizing some norm of the covariance matrix of the prediction error $\mathbf{e}_t$. Because of the recursive nature of the equation, the minimization must be solved using non-linear optimization. As it appears, the $\mathbf{A}_i$ matrices, denoted as the Auto Regressive (AR) parameters, and the $\mathbf{B}_i$ matrices, denoted as the Moving Average (MA) parameters, both contain np^2 real scalar parameters. This corresponds to a total of $2np^2$ real numbers that has to be estimated when calibrating the model. If $\mathbf{y}_t$ is the response of a structure, then poles appear in complex conjugate pairs, and thus, in general np must be a multiplum of two, and the number of structural modes is then $np/2$.

For a realistic case, the number of parameters might be so large, that problems estimating the parameters by non-linear optimization becomes time consuming. For a typical number of channels, say 16 channels, and for $n = 4$, the number of modes appearing in complex conjugate pairs (including noise modes) becomes 32, and the total number of parameters is 2048. This is a large set of parameters to be estimated, and this results in well known problems with calculation time, computer memory management and convergence. Thus, when using ARMA models for modal extraction, a simple way of reducing the number of parameters is needed.

Performing the optimization in the ARMA domain, i.e. using the $\mathbf{A}_i, \mathbf{B}_i$ matrices as the parameter set, there is no simple way of reducing the number of parameters. The $\mathbf{A}_i$ matrices are directly associated with the mode shapes and poles. The np^2 AR parameters correspond exactly to $np/2$ natural frequencies, damping ratios and scaled mode shapes. However, any element of the AR coefficient matrices influences all modal parameters, and thus, the AR parameter set cannot be reduced. The MA parameters $\mathbf{B}_i$ take care of the covariance equivalence and the noise modeling, however, at the same time these parameters have modal relation. They define the residues, and again, any element in the MA coefficient matrices is influencing all residues. The conclusion is, that the ARMA set of parameters does not allow a reduction of the parameter set to the modal parameters of interest.

However, if a transformation to and from the modal domain can be formulated, the optimization can be performed in modal domain, and thus, the parameter set can be reduced to the modal parameters of interest.

FROM ARMA TO MODAL DOMAIN

One of the difficulties of the ARMA model as formulated in Equation (1) is the high and changing order of the difference equation. This difficulty is removed by using a stochastic state space representation, Kailath [1980] and Aoki [1990]

$$\mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t + \mathbf{B}\mathbf{e}_t \quad (3)$$

$$\mathbf{y}_t = \mathbf{C}\mathbf{x}_t + \mathbf{e}_t \quad (4)$$

where $\mathbf{x}_t$ is the state vector whose elements depends on the actual realization. An example of a realization of a vector ARMA model is given in Andersen and Brincker [1999, 1].

Now, introducing the shift operator q defined by $q\mathbf{x}_t = \mathbf{x}_{t+1}$, the Equation (3) can now be written

$$\mathbf{I}q\mathbf{x}_t = \mathbf{A}\mathbf{x}_t + \mathbf{B}\mathbf{e}_t \quad (5)$$

or

$$(\mathbf{I}q - \mathbf{A})\mathbf{x}_t = \mathbf{B}\mathbf{e}_t \quad (6)$$

substituting this result for $\mathbf{x}_t$ into Equation (4) yields

$$\mathbf{y}_t = \mathbf{C}(\mathbf{I}q - \mathbf{A})^{-1}\mathbf{B}\mathbf{e}_t + \mathbf{e}_t \quad (7)$$

where $\mathbf{I}$ is the identity matrix. From this result the transfer function of the ARMA model, given by $\mathbf{y}_t = \mathbf{H}(q)\mathbf{e}_t$, can be obtained

$$\mathbf{H}(q) = \mathbf{C}(\mathbf{I}q - \mathbf{A})^{-1}\mathbf{B} + \mathbf{I} \quad (8)$$

the next step is to decompose the transfer function. This is done by a modal decomposition of the $\mathbf{A}$ matrix

$$\mathbf{A} = \mathbf{V}[\mu_i]\mathbf{V}^{-1} \quad (9)$$

where $\mathbf{V}$ is the eigenvector matrix, and $[\mu_i]$ is a diagonal matrix holding the eigenvalues. Substituting this into Equation (8) we get

$$\begin{aligned}\mathbf{H}(q) &= \mathbf{C}(\mathbf{I}q - \mathbf{V}[\mu_i]\mathbf{V}^{-1})^{-1}\mathbf{B} + \mathbf{I} \\ &= \mathbf{C}\mathbf{V}[(q-\mu_i)^{-1}]\mathbf{V}^{-1}\mathbf{B} + \mathbf{I} \\ &= \mathbf{\Phi}[(q-\mu_i)^{-1}]\mathbf{\Psi} + \mathbf{I} \quad (10)\end{aligned}$$

From this equation it appears, that $\mathbf{\Phi} = [\varphi_1, \varphi_2, ..]$ is the matrix holding the left mode shapes φ_i of the system as column vectors and similarly $\mathbf{\Psi} = [\psi_1^T, \psi_2^T, ..]^T$ is holding the right mode shapes ψ_i as row vectors. Note that both $\mathbf{\Phi}$ and $\mathbf{\Psi}^T$ are complex valued $p \times np$ matrices.

The left mode shapes is the observable part and correspond to the physical mode shapes. Both the left mode shapes and the right mode shapes define the residues. The eigenvalues μ are in fact the discrete poles of the system, this is easily recognized doing a partial fractional expansion of the transfer function

$$\begin{aligned} \mathbf{H}(q) &= \sum_{i=1}^{np} \frac{\varphi_i \psi_i}{q - \mu_i} + \mathbf{I} \\ &= \sum_{i=1}^{np} \frac{\mathbf{R}_i}{q - \mu_i} + \mathbf{I} \end{aligned} \tag{11}$$

where $\mathbf{R}_i$ is the residue matrix for i'th mode, a complex valued $p \times p$ matrix of rank one.

This solution defines the parameter set in the modal domain. If one wants to optimize on all parameters associated with one mode, the parameter set is the corresponding pole μ and the left and the right mode shape φ, ψ. If the noise modeling is excluded, only the pole and the left mode shape φ is included, and finally, if only the natural frequency and the damping is needed, the set is just the pole μ. Note, that the discrete pole μ is related to the pole λ in continuous time by

$$\mu = e^{\lambda \Delta t} \tag{12}$$

and that the relation between the continuos time pole and the corresponding natural frequency f and the damping ratio ζ is given by

$$\lambda = -2\pi f \zeta \pm i 2\pi f \sqrt{1 - \zeta^2} \tag{13}$$

This transformation explains the physical meaning of the ARMA model, it relates the AR and MA matrices to the structural system under consideration. Doing optimization however, this transformation does not help much. It defines the parameter set in modal domain, but does not provide the right tool for the optimization algorithm. Optimizing on the parameter set in modal domain, the inverse transformation is needed

FROM MODAL TO ARMA DOMAIN

Starting in modal domain we have the mode shape matrices $\mathbf{\Phi}$, $\mathbf{\Psi}$ and the diagonal eigenvalue matrix $[\mu_i]$. The modal set constitutes the transfer function as decribed by Equation (10). Comparing with Equation (8), which defines the transfer function in the state spate matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$, it is clear that the following state space realization exists

$$\begin{aligned} \mathbf{x}_{t+1} &= [\mu_i]\mathbf{x}_t + \mathbf{\Psi}\mathbf{e}_t & (14) \\ \mathbf{y}_t &= \mathbf{\Phi}\mathbf{x}_t + \mathbf{e}_t & (15) \end{aligned}$$

In this formulation, the state vector $\mathbf{x}$ is transformed by the eigenvector matrix $\mathbf{V}$. The formulation can be converted to the ARMA format by comparing solutions in modal and ARMA domain.

The first step is to compare the solutions of the homogenous equations. A solution to Equations (14) and (15) for $\mathbf{e}_t = 0$ is easily obtained by recursion

$$\mathbf{y}_{t+l} = \mathbf{\Phi}[\mu_i]^l \mathbf{x}_t \tag{16}$$

Substituting this solution into the corresponding solution to the homogenous part of the ARMA model given by Equation (1) and dividing on both sides by $\mathbf{x}_t$ yields the following equation

$$\mathbf{\Phi}[\mu_i]^n + \mathbf{A}_1\mathbf{\Phi}[\mu_i]^{n-1} + ... + \mathbf{A}_{n-1}\mathbf{\Phi}[\mu_i] + \mathbf{A}_n\mathbf{\Phi} = 0 \tag{17}$$

which can be solved for the AR coefficient matrices

$$[\mathbf{A}_1 \quad \mathbf{A}_2 \quad \ldots \quad \mathbf{A}_n] = -\mathbf{\Phi}[\mu_i]^n \begin{bmatrix} \mathbf{\Phi}[\mu_i]^{n-1} \\ \\ \cdot \\ \mathbf{\Phi}[\mu_i] \\ \mathbf{\Phi} \end{bmatrix}^{-1} \tag{18}$$

Without any proof the readers attention is drawn to the fact that in this solution the AR matrices always comes out as $p \times p$ real valued matrices.

The MA part is obtained by considering the general solution to Equations (14) and (15) found by recursive substitution

$$\mathbf{y}_{t+l} = \mathbf{\Phi}[\mu_i]^l \mathbf{x}_t + \sum_{j=1}^{l} \mathbf{\Phi}[\mu_i]^{l-j} \mathbf{\Psi} \mathbf{e}_{t+j-1} + \mathbf{e}_{t+l} \quad (19)$$

By stacking the general solution for $\mathbf{y}_t$ to $\mathbf{y}_{t-n+1}$ together with the observation equation (15) the following set of equations is obtained

$$\begin{bmatrix} \mathbf{y}_t \\ \mathbf{y}_{t-1} \\ \\ . \\ . \\ \mathbf{y}_{t-n+1} \\ \mathbf{y}_{t-n} \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}[\mu_i]^n \\ \mathbf{\Phi}[\mu_i]^{n-1} \\ \\ . \\ . \\ \mathbf{\Phi}[\mu_i] \\ \mathbf{\Phi} \end{bmatrix} \mathbf{x}_{t-n} + \begin{bmatrix} \mathbf{I} & \dots & \mathbf{\Phi}[\mu_i]^{n-2}\mathbf{\Psi} & \mathbf{\Phi}[\mu_i]^{n-1}\mathbf{\Psi} \\ 0 & \dots & \mathbf{\Phi}[\mu_i]^{n-3}\mathbf{\Psi} & \mathbf{\Phi}[\mu_i]^{n-2}\mathbf{\Psi} \\ . & \dots & & \\ . & \dots & . & . \\ . & \dots & . & . \\ 0 & \dots & \mathbf{I} & \mathbf{\Phi\Psi} \\ 0 & \dots & 0 & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{e}_t \\ \mathbf{e}_{t-1} \\ \\ \\ \\ \mathbf{e}_{t-n+1} \\ \mathbf{e}_{t-n} \end{bmatrix} \quad (20)$$

Now, multiplying this equation from left by the matrix $[\mathbf{I}\,\mathbf{A}_1\,\mathbf{A}_2 \dots \mathbf{A}_n]$ then the left-hand side equals the Auto Regressive part of Equation (1). According to Equation (17) the first part of the right-hand side of the equation vanish, and the last part then defines the Moving Average part of Equation (1), thus

$$[\mathbf{I} \quad \mathbf{B}_1 \quad \mathbf{B}_2 \quad \dots \quad \mathbf{B}_n] = [\mathbf{I}\,\mathbf{A}_1\,\mathbf{A}_2 \dots \mathbf{A}_n] \begin{bmatrix} \mathbf{I} & \dots & \mathbf{\Phi}[\mu_i]^{n-2}\mathbf{\Psi} & \mathbf{\Phi}[\mu_i]^{n-1}\mathbf{\Psi} \\ 0 & \dots & \mathbf{\Phi}[\mu_i]^{n-3}\mathbf{\Psi} & \mathbf{\Phi}[\mu_i]^{n-2}\mathbf{\Psi} \\ & \dots & & \\ & \dots & & \\ . & \dots & . & . \\ 0 & \dots & \mathbf{I} & \mathbf{\Phi\Psi} \\ 0 & \dots & 0 & \mathbf{I} \end{bmatrix} \quad (21)$$

Now the basis for optimization in modal domain is established. From the full set of modal parameters a subset is chosen for optimization. From the full set of parameters the ARMA matrices are obtained from the above equations, and the parameters in the optimization set can then in every optimization step be changed in order to minimize the prediction error. A technique of minimizing the prediction error is described in Andersen and Brincker [1999, 1].

OUTLINE OF ESTIMATION ALGORITHM

The idea of the algorithm is already explained above, however a short outline of the algorithm is given in the following.

The estimation procedure is divided in to three major steps: Initialization, minimization and uncertainty estimation.

1. Initialization. Step 1.a: In this step an initial estimate for the ARMA model is provided. Any initial estimate can be used that provide the needed modal information (poles and left and right mode shapes). *Step 1.b*: In this step the modal parameters in the optimization parameter set is selected.

2. Minimization by recursion. Step 2.a: The first step (not to be performed the first time) is to detect if the optimization parameter vector has caused a significant decrease of the measure of the prediction errors. To do this, the full modal parameter set is transformed to ARMA domain, the prediction erros are determined and the measure of the prediction errors is calculated. If there is no significant change of this measure step 2 is terminated. *Step 2.b*: Based on the modal parameter optimization set the search gradient is constructed. *Step 2.c*: Based on the search gradient the modal parameter optimization set is updated. Continue the recursion by repeating from *Step 2.a.*

3. Uncertainty estimation. Using PEM estimation allows for estimation of the covariance matrix of the optimization parameter set, Ljung [1987] and Andersen and Brincker [1999, 1]. Assuming that the subset of parameters consist of all uncertain parameters in the model, the covariance matrix of the parameter subset can be estimated. When using traditional ARMA model calibration, where the calibration is performed in ARMA domain, in many cases numerical difficulties or additional uncertainties are introduced by transforming the parameter covariance matrix from ARMA domain to modal domain. In this case however, where the parameter vector contain the modal parameters, the modal parameter covariance matrix is estimated directly

CONCLUSIONS

A new technique for PEM estimation of ARMA models for structural ambient responses has been introduced. Based on analytical solutions for the transformation between ARMA domain and modal domain, the optimization is performed in modal domain. The parameter set contain the modal parameters, or if only some of the modal parameters are of interest, a subset of the modal parameters. The set of modal parameters consists of the poles (natural frequencies and damping ratios), the left mode shapes (scaled mode shapes) and the right mode shapes. The technique allows for a reduction of the parameter set to a minimum set consisting only of the modal parameters of interest. This strongly reduces the well known problems often observed in practice concerning long estimation times, memory management problems and convergence problems. The technique also allows for a more reliable and more accurate estimation of the modal parameter covariance matrix. Applications of the technique can be found in Andersen et al. [1999, 2] and Peeters et al. [1999, 3].

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