

# A Way of Getting Scaled Mode Shapes in Output Only Modal Testing

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## Abstract

*In this paper some further work is done following the idea introduced by Parloo et al [2] where they proposed that the scaling factor should be estimated by repeated testing introducing mass changes in different points on the structure. In this paper the approximate formula for determination of the scaling factor based on the frequency shift when introducing mass changes on the structure is derived directly from the governing equation of motion. Further the error sources are studied and a new formula introducing less approximation errors on the scaling factor is proposed. Using this new formula scaling factors can be estimated from relative large frequency shifts without introducing any approximation errors. Further it is explained how testing should be performed in order to significantly reduce approximation errors due to mode shape changes and random errors due to uncertainty on the mode shape values. It turns out that if the mass changes are well distributed over the structure, then both random errors and the approximation errors will be minimized.*

## Notation

Mass matrix	$\mathbf{M}$
Stiffness matrix	$\mathbf{K}$
Un-scaled mode shape	$\phi$
Scaled mode shape	$\psi$
Natural frequency	$\omega$
Standard deviation	$\sigma$
Number of DOF's	$N$

## 1. Introduction

Output-only modal testing and analysis is becoming more and more popular due to the clear advantages of the technology: The testing is easier, the technology is applicable to a wider range of the structures, it has a wider range of potential applications since the actual responses are stored and can be used for instance fatigue analysis and vibration level estimation, and also some times the

technology gives better and more reliable results in cases where the actual loading conditions and operating conditions are important for the structural response.

However the technology is still in the early stages of development, and many problems still remain unsolved or partly unsolved. One the important remaining problems is the problem of mode shape scaling. If the identified modal model is going to be used for structural response simulation or for structural modification, then the scaling factors of the mode shapes must be known. Also in health monitoring applications and in cases where damage is to be identified, the scaling factors might be useful.

Recently some suggestion has been given in the literature for solving this problem. One solution has been suggested by Bernal and Gunes [1] based on the assumption that partition of the inverse of the mass matrix associated with the measured coordinates is diagonal. However, the approach gives exact answers only when there is a full set of modes, and robustness for a truncated modal space has not been demonstrated. Recently Parloo et al [2] have published a new approach based on a more extensive testing procedure that involves repeated testing where mass changes are introduced in the points of the structure where the mode shape is known. This approach seems more appealing, since to scale a certain mode, only that particular mode has to be known.

Parloo et al derived an approximate formula for the determination of the scaling factor from some basic sensitivity relations in linear dynamics. In this paper however we will derive the same approximate formula directly from the equations of motion, an analysis of the random errors and of the errors introduced by the approximations is presented, and finally a proposal for a better formula introducing significantly less approximations errors will be given.

## 2. Approximate scaling from mass change effects

The approximate equation that relates mass change, frequency change and the unknown scaling factor can be

easily derived from the basic equations of motion. The classical eigenvalue equation in case of no damping is

$$(1) \quad \mathbf{M}\boldsymbol{\varphi}_1\omega_1^2 = \mathbf{K}\boldsymbol{\varphi}_1$$

where  $\boldsymbol{\varphi}_1$  is the mode shape and  $\omega_1$  is the natural frequency of any of the modes of the problem related to the mass matrix  $\mathbf{M}$ . Now, if we make a mass change so that the mass matrix becomes  $\mathbf{M} + \Delta\mathbf{M}$ , then the eigenvalue equation becomes

$$(2) \quad (\mathbf{M} + \Delta\mathbf{M})\boldsymbol{\varphi}_2\omega_2^2 = \mathbf{K}\boldsymbol{\varphi}_2$$

and  $\boldsymbol{\varphi}_2, \omega_2$  is now the modal parameters of the modified problem.

Subtracting equation (1) and (2) then gives us the equation to approximate to obtain an equation for the unknown scaling factor

$$(3) \quad \mathbf{M}(\boldsymbol{\varphi}_1\omega_1^2 - \boldsymbol{\varphi}_2\omega_2^2) = \Delta\mathbf{M}\boldsymbol{\varphi}_2\omega_2^2 + \mathbf{K}(\boldsymbol{\varphi}_2 - \boldsymbol{\varphi}_1)$$

If we now assume that the mass change is so small that the mode shapes does not change significantly, i.e.

$$(4) \quad \boldsymbol{\varphi}_1 \cong \boldsymbol{\varphi}_2 = \boldsymbol{\varphi}$$

and that the change of natural frequency is so small that with good approximation

$$(5) \quad \omega_1^2 - \omega_2^2 \cong 2\omega\Delta\omega$$

where  $\Delta\omega = \omega_1 - \omega_2$  and  $\omega = (\omega_1 + \omega_2)/2$ , then we obtain the equation

$$(6) \quad 2\mathbf{M}\boldsymbol{\varphi} \frac{\Delta\omega}{\omega} = \Delta\mathbf{M}\boldsymbol{\varphi}$$

This equation holds for any mode shape scaling factor. Usually in output-only modal analysis the mode shapes are just scaled to unity, i.e.  $\boldsymbol{\varphi}^T\boldsymbol{\varphi} = 1$ , and usually in traditional modal analysis is preferred a scaling so that

$$(7) \quad \boldsymbol{\psi}^T\mathbf{M}\boldsymbol{\psi} = 1$$

The desired scaling factor is the factor that relates the scaled and the un-scaled mode shapes

$$(8) \quad \boldsymbol{\psi} = \alpha\boldsymbol{\varphi}$$

Now using the scaled mode shapes in Eq. (4) and combining with Eq. (6), we directly obtain by multiplying with the same mode shape from the left

$$(9) \quad 2 \frac{\Delta\omega}{\omega} = \alpha^2 \boldsymbol{\varphi}^T \Delta\mathbf{M}\boldsymbol{\varphi}$$

and the unknown scaling factor is obtained as

$$(10) \quad \alpha = \sqrt{\frac{2\Delta\omega}{\boldsymbol{\varphi}^T \Delta\mathbf{M}\boldsymbol{\varphi}}}$$

Which is the result obtained by in Parloo et al (equation (10) in [2]). The matrix  $\Delta\mathbf{M}$  would normally be a diagonal matrix corresponding to the case where the mass changes are placed only at the points where the mode shapes are known. If this is case, and if all mass changes  $\Delta m$  at the individual degrees of freedom are the same, then the mass change matrix can be written  $\Delta\mathbf{M} = \Delta m\mathbf{D}$ , where the matrix  $\mathbf{D}$  is a diagonal matrix containing only ones in the diagonal. Then equation (8) becomes

$$(11) \quad \alpha = \sqrt{\frac{2\Delta\omega}{\omega\Delta m Z}}$$

$$(12) \quad Z = \boldsymbol{\varphi}^T \mathbf{D}\boldsymbol{\varphi}$$

In the matrix  $\mathbf{D}$  there is a one in the diagonal every time an additional mass is placed in that DOF, and if a mass is removed, there is a minus one. Thus, if a set of masses are placed at the structure, the number  $Z$  is a positive, and if the mass is removed, then the number  $Z$  is negative. If the un-scaled mode shapes are scaled to unity, then the absolute value of  $Z$  is between zero and one, and in the case the same mass is placed in all DOF's at the structure, then  $Z$  becomes equal to one.

### 3. Uncertainty on the estimated scaling factors

Assuming that the uncertainty  $\sigma_\alpha$  on  $\alpha$  is controlled by the uncertainty  $\sigma_{\Delta\omega}$  on the frequency shift  $\Delta\omega$  and by the uncertainty  $\sigma_Z$  on the inner product  $Z$ , then linearization of equation (9) and assuming independent stochastic variables leads to the following approximate equation for the relative variance on the scaling factor

$$(13) \quad \frac{\sigma_\alpha^2}{\alpha^2} = \frac{1}{4} \frac{\sigma_{\Delta\omega}^2}{\Delta\omega^2} + \frac{1}{4} \frac{\sigma_Z^2}{Z^2}$$

The uncertainty on the inner product  $Z$  can be obtained in a similar way to

$$(14) \quad \sigma_Z^2 = 4\sigma_\varphi^2 \boldsymbol{\varphi}^T \mathbf{D}\boldsymbol{\varphi}$$

where the uncertainty on the mode shape values are assumed all to be equal to  $\sigma_\varphi$ . Combining equation (13) and (14) then yields the final result

$$(15) \quad \frac{\sigma_\alpha^2}{\alpha^2} = \frac{1}{4} \frac{\sigma_{\Delta\omega}^2}{\Delta\omega^2} + \frac{\sigma_\varphi^2}{\boldsymbol{\varphi}^T \mathbf{D}\boldsymbol{\varphi}}$$

From this formula it is clearly observed, that it is important to keep the relative uncertainty of the frequency shift down to a

reasonable value, the same can be said about the mode shape values, however, for the mode shapes it is also important to use as many points as possible for the mass change. Thus, it can be concluded, that in order to minimize the uncertainty on the scaling factor, the mass changes should be distributed over as many degrees of freedom as possible, and the mass change should be large enough to allow for a reasonably low uncertainty of the natural frequency shift. However, the mass changes should be small enough for the approximations given by equation (5) and (6) to be valid. A further analysis of the errors will be given in the next sections.

#### 4. Illustration of approximation errors

The errors introduced by the approximate nature of equations (4) and (5) has been studied by performing simulation on a simple dynamic system

$$(16) \quad \mathbf{M} = m \begin{bmatrix} 1 & 0 & \cdot \\ 0 & 1 & \\ \cdot & & \cdot \\ & & & 1 \end{bmatrix} \quad \mathbf{K} = k \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & & & \\ & & \cdot & & -1 \\ & & & \cdot & \\ & & & & -1 & 2 \end{bmatrix}$$

The system had 20 degrees of freedom and only the first 5 modes were used in the investigation. The exact scaling factor  $\alpha$  were calculated using equation (7) and (8). Then a mass change was introduced, and equation (10) was then used to obtain an approximate value  $\alpha'$  for the scaling factor. The error on scaling factor was then calculated as  $\alpha/\alpha'-1$ . Figure 1 shows the error on the scaling factor for the first mode of the system when performing a mass change in one DOF (top plot) and five DOF's (bottom plot).

As it is observed from Figure 1, when shifting the mass changes over the structure, for some cases the error is negative, and for some cases it is positive. However, it turns out that the largest error is positive and corresponds to the upper envelope of the curves (the approximate linear upper bound).

Taking the envelopes for the first five modes gives the results shown in Figure 2. As it is seen from these results, when the mass change is oddly distributed (mass change in 5 DOF's), the error is significantly dependent on the mode, and when the mass changes becomes more and more evenly distributed, the difference in error between the individual modes tends to vanish.

This is to be expected from theoretical considerations. When the mass change is evenly distributed so that

$$(17) \quad \Delta \mathbf{M} = \beta \mathbf{M}$$

then mode shapes does not change by introducing the additional masses, and thus, the assumption giving by equation (4) is satisfied and does not contribute to the approximation error.

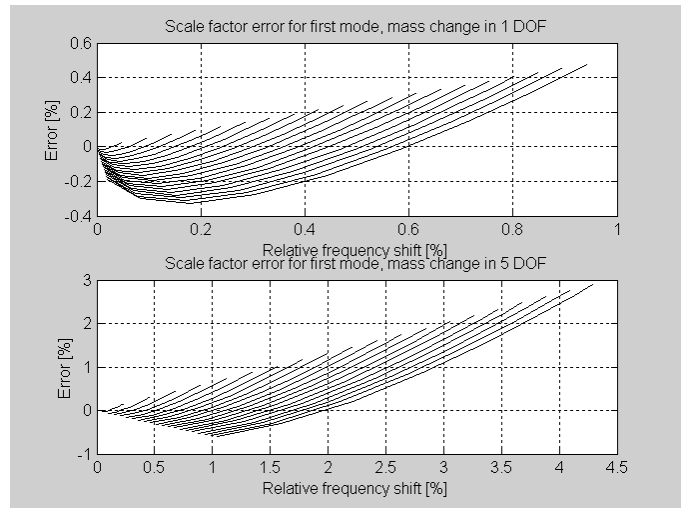


Figure 1. Top: Error on scaling factor when using equation (10) for a mass change in one DOF. Each curve shows the influence of the mass change moving over the structure, the different curves shows the influence of a mass change from 1-20 % of the mass of the individual DOF's. Bottom: Same as the top, but now with mass change in 5 DOF's, however, since the total mass change is five times larger, the frequency shift is larger.

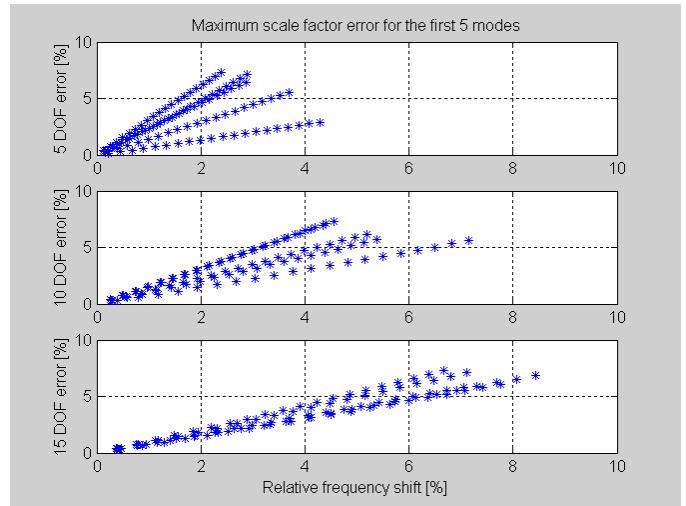


Figure 2. Maximum errors when using equation (10) on the first five modes of the investigated system when changing the mass at the individual DOF's from 1-20 %. Top: mass changes at 5 DOF's, middle: mass changes at 10 DOF's, bottom: mass changes at 15 DOF's.

#### 5. A better formula for the scaling factor estimation

As it appears from the previous analysis, the errors on the scaling factor when using the approach as proposed by Parloo et al have two sources of error both contributing significantly; a) one error source due the mode shape changes when masses are added to the structure (equation (4)), and b) one error source due to the approximation given by equation (5). The first error source can be minimized by

distributing the mass changes according to equation (17), however, the latter error source is inherent to application of equation (10) and cannot be altered as long as this approach is used.

However, there is no reason to perform the approximation given by equation (5). The resulting equation when only using the approximation (4) and not approximation (5) is

$$(18) \quad \alpha = \sqrt{\frac{\omega_1^2 - \omega_2^2}{\omega_2^2 \boldsymbol{\varphi}^T \boldsymbol{\Delta} \mathbf{M} \boldsymbol{\varphi}}$$

When using this formula, no approximation is introduced due to final values of the frequency shift, and thus, large frequency shifts can be applied without introducing any error on the scaling factor. The error reduction when using equation (18) in stead of equation (10) is illustrated in Figure 3.

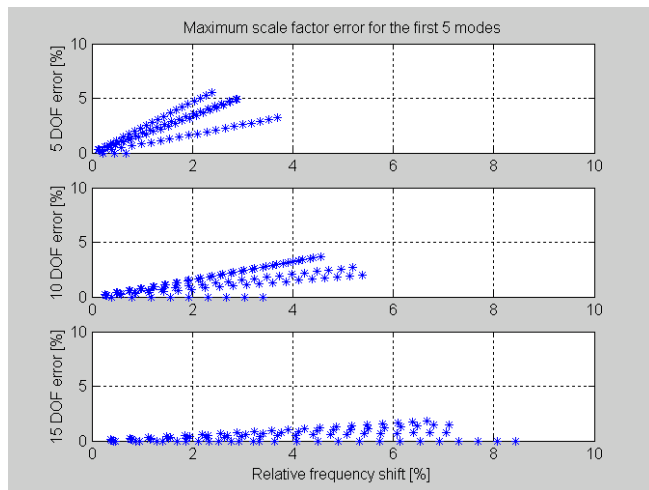


Figure 3. Maximum errors when using equation (18) on the first five modes of the investigated system when changing the mass at the individual DOF's from 1-20 %. Top: mass changes at 5 DOF's, middle: mass changes at 10 DOF's, bottom: mass changes at 15 DOF's.

## 6. Error reduction strategies

It can be concluded that the approximate equation (10) should not be used for mode shape scaling since it introduces significant and unnecessary errors on the scaling factor. In stead equation (18) should be used. When using equation (18) three sources of errors are still present

- a) Systematic errors due to mode shape changes
- b) Random errors on the frequency shift
- c) Random errors on the mode shape

As it has been shown above, the best way to reduce the errors introduced by changes of the mode shape is to try to perform the mass changes in such a way that all DOF's have the same relative mass change. This approach assumes that before performing the test a reasonable estimate of the mass distribution can be obtained.

The error on the frequency shift can be dealt with by performing good tests and good analysis in order to keep the uncertainty on the natural frequency identification down and making sure that the frequency shift is significantly larger than the uncertainty. A good natural frequency identification can usually be performed with a relative accuracy of the order of  $10^{-3}$ . Thus if we make sure that the frequency shift is at least 1 % of the natural frequency, then the contribution to the relative error on the scaling will be within 10 %. This does not seem satisfactory, and thus higher identification accuracy has to be achieved and/or larger mass changed must be applied. However, using equation (18) in stead of equation (10) will allow for the application of large frequency shifts without any error introduction.

The influence of the random errors on the mode shape can be reduced by introducing the mass changes in as many DOF's as possible. If we only perform mass change is one DOF, then the error contribution is

$$(19) \quad \frac{\sigma_{\alpha}}{\alpha} = \frac{\sigma_{\varphi}}{\varphi_k}$$

but if we introduce mass changes in all the DOF's, then – if we assume that the un-scaled mode shapes are scaled to unity - the contribution becomes

$$(20) \quad \frac{\sigma_{\alpha}}{\alpha} = \sigma_{\varphi}$$

If we know the mode shape in  $N$  DOF's, the mode shape values will be of the order  $\gamma = \sqrt{1/N}$  and thus, if we have estimated mode shapes within a relative accuracy of  $\varepsilon$ , then the uncertainty on the individual mode shape values will be of the order  $\varepsilon^2 / N$  and then

$$(21) \quad \frac{\sigma_{\alpha}}{\alpha} \cong \varepsilon / \sqrt{N}$$

Thus, if a mode shape is known in say 100 points, and if the uncertainty on the mode shape is smaller then 10 %, then introducing mass changes in all 100 points, the uncertainty on the mode shape will only contribute to the random error on the scaling factor with less than 1 %.

The main conclusion is that mode shape scaling using the strategy proposed in this paper should be performed by using mass changes that are as well distributed as possible, and in such a way that the resulting changes of the natural frequencies are large compared to the uncertainty on the natural frequencies.

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