Uncertainty Quantification for Stochastic Subspace Identification of Multi-Setup Measurements

Michael Döhler, Xuan-Binh Lam, Laurent Mevel
INRIA Centre Rennes – Bretagne Atlantique, Rennes, France

Palle Andersen
Structural Vibration Solutions, Aalborg, Denmark

ABSTRACT: In Operational Modal Analysis, the modal parameters (natural frequencies, damping ratios and mode shapes) obtained from Stochastic Subspace Identification (SSI) of a structure, are afflicted with statistical uncertainty. For evaluating the quality of the obtained results it is essential to know the appropriate confidence intervals of these figures. In this paper we present algorithms that automatically compute the confidence intervals of modal parameters obtained from covariance-driven and data-driven SSI of a structure based on vibration measurements. These algorithms are adapted to handle data from different measurements of a structure, where roving sensors are moved from one measurement setup to another, while some reference sensors stay fixed throughout all the measurements. In this case, the different ambient excitations of the structure between the measurements have to be taken into account. With these new algorithms, confidence intervals of the modal parameters of some relevant industrial example are computed.

1 INTRODUCTION

Subspace-based system identification methods have been proven efficient for the identification of the eigenstructure of linear multivariable systems. An important application of these methods is Operational Modal Analysis, where the modal parameters (frequencies, damping ratios and mode shapes) are identified of mechanical, civil or aeronautical structures subject to uncontrolled, unmeasured and nonstationary excitation.

To obtain vibration measurements at many coordinates of a structure with only few sensors, it is common practice to use multiple sensor setups for the measurements. For these *multi-setup measurements*, some of the sensors, the so-called reference sensors, stay fixed throughout all the setups, while the other sensors are moved from setup to setup. By merging in some way the corresponding data while taking into account possible different ambient excitations between the measurements, this allows to perform modal identification as if there was a large number of sensors. A global merging approach was proposed in (Mevel et al. 2002a, b), where the data from the different setups is normalized and merged first, followed by a global system identification step. Recently, this approach was generalized to a large range of stochastic subspace algorithms in (Döhler and Mevel 2010, 2011b), including covariance-driven and data-driven algorithms such as the Unweighted Principal Component algorithm (UPC, Van Overschee and De Moor 1996).

The obtained modal parameters are afflicted with statistical uncertainty due to measurement noise, nonstationary and colored excitation noise, model order truncation and many other sources. To evaluate the quality of the estimated modal parameters, it is thus essential to quantify this uncertainty. In (Reynders et al. 2008), this was done for the modal parameter estimation with covariance-driven SSI. In (Döhler et al. 2011), the uncertainty of modal parameters estimated with the data-driven UPC algorithm was computed. In this paper, the uncertainty quantification is generalized to stochastic subspace identification using multi-setup measurements.

2 MULTI-SETUP STOCHASTIC SUBSPACE IDENTIFICATION

2.1 Models and Parameters

The behaviour of a mechanical system is assumed to be described by a stationary linear dynamical system

$$M\ddot{Z}(t) + CZ(t) + KZ(t) = v(t), \quad Y(t) = LZ(t), \tag{1}$$

where t denotes continous time, M, C and K are the mass, damping and stiffness matrices, high-dimensional vector Z collects the displacements of the degrees of freedom of the structure, the non-measured external force v modelled as non-stationary Gaussian white noise, the measurments are collected in the vector Y and matrix L indicates the sensor locations.

The eigenstructure of (1) with the modes μ and mode shapes φ_μ is a solution of

$$\det(\mu^2 M + \mu C + K) = 0, \quad (\mu^2 M + \mu C + K)\phi_\mu = 0, \quad \varphi_\mu = L\phi_\mu . \tag{2}$$

Sampling model (1) at some rate $1/\tau$ yields the discrete model in state-space form

$$X_{k+1} = FX_k + V_{k+1}, \quad Y_k = HX_k, \tag{3}$$

whose eigenstructure is given by

$$\det(F - \lambda I) = 0, \quad (F - \lambda I)\phi_\lambda = 0, \quad \varphi_\lambda = H\phi_\lambda . \tag{4}$$

Then, the eigenstructure of the continous system (1) is related to the eigenstructure of the discrete system (3) by

$$e^{\tau\mu} = \lambda, \quad \varphi_\mu = \varphi_\lambda . \tag{5}$$

The collection of modes and mode shapes $(\lambda,\varphi_\lambda)$ is a canonical parameterization of system (3). From the eigenvalues μ the natural frequencies f and damping ratios d with

$$f = \mathrm{Im}(\mu)/(2\pi), \; \mathrm{d} = -\mathrm{Re}(\mu)/|\mu|. \tag{6}$$

are retrieved.

2.2 Single-Setup Stochastic Subspace Identification

To obtain the modal parameters (frequencies, damping ratios and mode shapes) from measurements $(Y_k)_{k=1,\ldots,N}$, the covariance-driven output-only subspace identification algorithm (Benveniste and Fuchs 1985, Peeters and De Roeck 1999) and the data-driven Unweighted Principal Component algorithm (Van Overschee and De Moor 1996, Peeters and De Roeck 1999) are used. They only differ in the computation the so-called subspace matrix **H**.

In the *covariance-driven SSI*, a block Hankel matrix **H** is filled with the correlation lags $R_i = \mathbf{E}(Y_k Y_{k-i}^T)$ of the output data

$$\mathbf{H} = \mathrm{Hank}(R_i) = \begin{pmatrix} R_0 & R_1 & \cdots & R_{q-1} \\ R_1 & R_2 & \cdots & R_q \\ \vdots & \vdots & \ddots & \vdots \\ R_p & R_{p+1} & \cdots & R_{p+q} \end{pmatrix}, \quad R_i = \sum_{k=i+1}^{N} Y_k Y_{k-i}^T . \tag{7}$$

In the *data-driven SSI*, first some data matrices $\mathbf{Y}^-$ and $\mathbf{Y}^+$ are built containg the data samples

$$\mathbf{Y}^{-}=\frac{1}{\sqrt{N}}\begin{pmatrix} Y_q & Y_{q+1} & \cdots & Y_{N+q-1} \\ Y_{q-1} & Y_q & \cdots & Y_{N+q-2} \\ \vdots & \vdots & \ddots & \vdots \\ Y_1 & Y_2 & \cdots & Y_N \end{pmatrix}, \mathbf{Y}^{+}=\frac{1}{\sqrt{N}}\begin{pmatrix} Y_{q+1} & Y_{q+2} & \cdots & Y_{N+q} \\ Y_{q+2} & Y_{q+3} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ Y_{q+p+1} & Y_{q+p+2} & \cdots & Y_{N+p+q} \end{pmatrix} \tag{8}$$

and the matrix **H** is obtained from the LQ decomposition of

$$\begin{pmatrix} \mathbf{Y}^{-} \\ \mathbf{Y}^{+} \end{pmatrix}=\begin{pmatrix} R_{11} & 0 \\ R_{21} & R_{22} \end{pmatrix}\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} \tag{9}$$

by $\mathbf{H} = R_{21}$.

For both algorithms, **H** possesses the factorization property

$$\mathbf{H}=\mathbf{O}\,\mathbf{X} \tag{10}$$

into observability matrix

$$\mathbf{O}=\begin{pmatrix} H \\ HF \\ \vdots \\ HF^{p} \end{pmatrix} \tag{11}$$

and some other matrix **X**, where **O** is obtained from **H** by an SVD and truncation at the desired model order:

$$\mathbf{H}=\begin{pmatrix} U_1 & U_0 \end{pmatrix}\begin{pmatrix} \Delta_1 & \\ & \Delta_0 \end{pmatrix}V^{T},\ \mathbf{O}=U_1\Delta_1^{1/2}. \tag{12}$$

From the observability matrix **O** the matrices H in the first block row and F from a least squares solution of

$$\overline{\mathbf{O}}\,F=\underline{\mathbf{O}}\quad\text{with}\quad \overline{\mathbf{O}}=\begin{pmatrix} H \\ HF \\ \vdots \\ HF^{p-1} \end{pmatrix},\ \underline{\mathbf{O}}=\begin{pmatrix} HF \\ HF^{2} \\ \vdots \\ HF^{p} \end{pmatrix} \tag{13}$$

are obtained. The eigenstructure $(\lambda,\varphi_\lambda)$ of the system (3) is then obtained in (4) and the corresponding frequencies and damping ratios in (5)-(6).

2.3 Multi-Setup Stochastic Subspace Identification

Instead of a single record for the output (Y_k) of the system (3), N_s records

$$\underbrace{\begin{pmatrix} Y_k^{(1,\mathrm{ref})} \\ Y_k^{(1,\mathrm{mov})} \end{pmatrix}}_{\text{Record 1}}\ \underbrace{\begin{pmatrix} Y_k^{(2,\mathrm{ref})} \\ Y_k^{(2,\mathrm{mov})} \end{pmatrix}}_{\text{Record 2}}\ \cdots\ \underbrace{\begin{pmatrix} Y_k^{(N_s,\mathrm{ref})} \\ Y_k^{(N_s,\mathrm{mov})} \end{pmatrix}}_{\text{Record } N_s} \tag{14}$$

are now available collected successively. Each record j contains data $Y_k^{(j,\mathrm{ref})}$ from a fixed reference sensor pool containing $r^{(\mathrm{ref})}$ sensors, and data $Y_k^{(j,\mathrm{mov})}$ from a moving sensor pool containing r_j sensors. As described in (Mevel et al. 2002a, b), to each record $j = 1, \ldots, N_s$ corresponds a state-space realization in the form

$$\begin{cases} X_{k+1}^{(j)} = F X_k^{(j)} + V_{k+1}^{(j)} \\ Y_k^{(j,\text{ref})} = H^{(\text{ref})} X_k^{(j)} \\ Y_k^{(j,\text{mov})} = H^{(j,\text{mov})} X_k^{(j)} \end{cases} \tag{15}$$

with a single state transition matrix F. Note that the unmeasured excitation $V^{(j)}$ can be different for each measurement j as the environmental conditions can slightly change between the measurements. Note also that the observation matrix $H^{(\text{ref})}$ is independent of the specific measurement setup if the reference sensors are the same throughout all setups $j = 1, \ldots, N_s$.

In (Mevel et al. 2002a, b) a method was described to normalize and merge data from multiple setups to obtain global modal parameters (natural frequencies, damping ratios, mode shapes). The normalization is important because the background excitation may differ between setups. As the normalization and merging step is done first, only one system identification of the global system is necessary, instead of having to do system identification of each setup separately and then merging the results. In (Döhler and Mevel 2010, 2011b) this global merging approach, which is valid for the covariance-driven SSI, was generalized to a large range of subspace algorithms and consists of the following steps:

a) For each setup j, build the matrix $\mathbf{H}^{(j)}$ from the data $Y_k^{(j,\text{ref})}$ and $Y_k^{(j,\text{mov})}$ (for covariance-driven SSI as in (5))
b) SVD of $\mathbf{H}^{(j)}$ as in (12) to get observability matrix $\mathbf{O}^{(j)}$
c) Separate $\mathbf{O}^{(j)}$ into $\mathbf{O}^{(j,\text{ref})}$ and $\mathbf{O}^{(j,\text{mov})}$, where the former contains the information w.r.t. the reference sensors ($H^{(\text{ref})}$) and the latter w.r.t. moving sensors ($H^{(j,\text{mov})}$).
d) Compute the "normalized" observability matrix part $\hat{\mathbf{O}}^{(j)} = \mathbf{O}^{(j,\text{mov})} (\mathbf{O}^{(j,\text{ref})})^{+} \mathbf{O}^{(1,\text{ref})}$, where $^{+}$ denotes the pseudoinverse
e) Interleave the matrices $\mathbf{O}^{(1,\text{ref})}$ and $\hat{\mathbf{O}}^{(j)}$, $j = 1,\ldots,N_s$, to a global observability matrix $\mathbf{O}^{(\text{all})}$, where H = $(H^{(\text{ref})T}\ \ H^{(1,\text{mov})T}\ \ H^{(2,\text{mov})T}\ \ \ldots\ \ H^{(N_s,\text{mov})T})^{T}$ in Definition (11)
f) Do global system identification of system (15) with SSI from Section 2.2 starting at Equation (13) using $\mathbf{O}^{(\text{all})}$

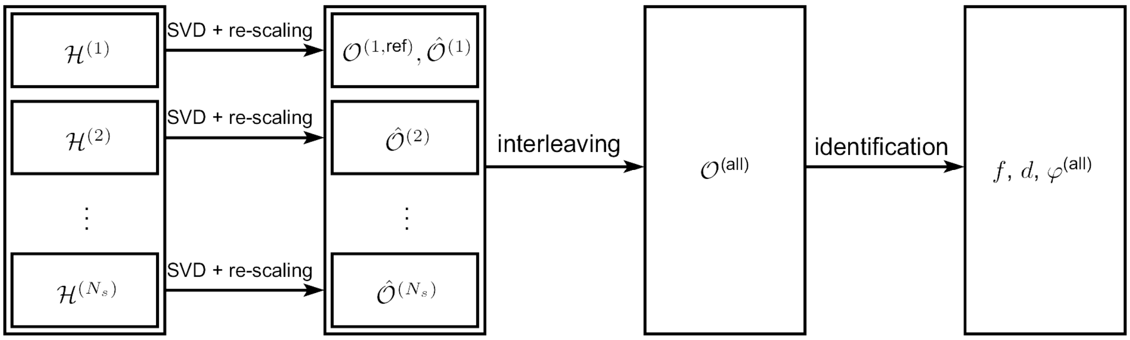

Figure 1: Multi-setup system identification with merging schme from (Döhler and Mevel 2010, 2011b).

3 CONFIDENCE INTERVALS ON MODAL PARAMETERS

3.1 Confidence Intervals on Modal Parameters in Single-Setup SSI

The statistical uncertainty of the obtained modal parameters at a chosen system order can be computed from the uncertainty of the system matrices, which depends on the covariance of the corresponding subspace matrix **H**. The latter can be evaluated by cutting the sensor data into blocks on which instances of the subspace matrix are computed, so this offers a possibility to compute the confidence intervals of the modal parameters at a certain system order without repeating the system identification on each data block. In (Reynders et al. 2008) this algorithm was described in detail for the covariance-driven SSI.

The uncertainty ΔF and ΔH of the system matrices F and H are connected to the uncertainty of the subspace matrix through a Jacobian matrix $J_{F,H}$ (Reynders et al. 2008):

$$\begin{bmatrix} \mathrm{vec}\,\Delta F \\ \mathrm{vec}\,\Delta H \end{bmatrix} = J_{F,H}\,\mathrm{vec}\Delta\mathbf{H} \tag{16}$$

Then, the uncertainty of the modal parameters (natural frequency f, damping ratio d and mode shape φ) is propagated

$$\Delta f_\lambda = J_{f_\lambda} \begin{bmatrix} \mathrm{vec}\,\Delta F \\ \mathrm{vec}\,\Delta H \end{bmatrix},\ \ \Delta d_\lambda = J_{d_\lambda} \begin{bmatrix} \mathrm{vec}\,\Delta F \\ \mathrm{vec}\,\Delta H \end{bmatrix},\ \Delta \varphi_\lambda = J_{\varphi_\lambda} \begin{bmatrix} \mathrm{vec}\,\Delta F \\ \mathrm{vec}\,\Delta H \end{bmatrix}, \tag{17}$$

with the Jacobians $J_{f\lambda}$, $J_{d\lambda}$ and $J_{\varphi\lambda}$ for each mode λ. Finally, the covariances of the modal parameters are obtained from (16) and (17) by

$$\mathrm{cov}(f_\lambda) = J_{f_\lambda} J_{F,H} \Sigma_\mathbf{H} J_{F,H}^T J_{f_\lambda}^T,\ \ \mathrm{cov}(d_\lambda) = J_{d_\lambda} J_{F,H} \Sigma_\mathbf{H} J_{F,H}^T J_{d_\lambda}^T,$$
$$\mathrm{cov}(\varphi_\lambda) = J_{\varphi_\lambda} J_{F,H} \Sigma_\mathbf{H} J_{F,H}^T J_{\varphi_\lambda}^T, \tag{18}$$

where $\Sigma_\mathbf{H}$ = cov(vec $\mathbf{H}$) is the covariance of the vectorized subspace matrix $\mathbf{H}$ which can be easily obtained from the output-only data. The computation of $\Sigma_\mathbf{H}$ was described for the covariance-driven SSI in (Reynders et al. 2008) and for the data-driven UPC algorithm in (Döhler and Mevel 2011a).

3.2 Confidence Intervals on Modal Parameters in Multi-Setup SSI

In the computation of the confidence intervals of the modal parameters on a single measurement setup in the previous section, matrices F and H only depend on one observability matrix $\mathbf{O}$ that is obtained from the subspace matrix $\mathbf{H}$. Now, doing system identification using multiple setups as in Section 2.3, the matrices F and H depend on several observability matrices $\mathbf{O}^{(1,\mathrm{ref})}$ and $\hat{\mathbf{O}}^{(j)}$, $j = 1,\ldots,N_s$, which are obtained from the subspace matrices $\mathbf{H}^{(j)}$, $j = 1,\ldots,N_s$, of each setup.

In (Lam et al. 2011) the computation of the confidence intervals for multi-setup SSI is explained in detail. It is based on the fact that measurements that are taken at different times are statistically independant from each other and hence the matrices $\mathbf{H}^{(j)}$, $j = 1,\ldots,N_s$, are statistically independent. Then, the covariance of the system matrices can be formulated as

$$\mathrm{cov}\left(\begin{bmatrix} \mathrm{vec}\,F \\ \mathrm{vec}\,H \end{bmatrix}\right) = \sum_{j=1}^{N_s} J_j \Sigma_{\mathbf{H}^{(j)}} J_j^T,\quad \text{where}\quad \Sigma_{\mathbf{H}^{(j)}} = \mathrm{cov}(\mathrm{vec}\,\mathbf{H}^{(j)}), \tag{19}$$

with sensitivities J_j of the system matrices with respect to the data of each setup j (Lam et al. 2011). Then, the covariances of the modal parameters are computed as in (18), where $J_{F,H}\Sigma_\mathbf{H}J_{F,H}{}^T$ is replaced by the sum in (19).

4 NUMERICAL RESULTS

We present the results on the multi-setup system identification and confidence interval compuation on a multilayer E-glass reinforced composite panel that is similar to the load carrying laminate in a wind turbine blade (Luczak et al. 2010). The nominal dimensions are 20x320x320 mm. Vibration measurements were taken using accelerometers in 3 measurement setups containing 14 moving sensors each and one setup containing 7 moving sensors, while one reference sensor stayed fixed during all the measurements.

In this section, we only present results obtained from the covariance-driven SSI. Results for data-driven SSI are very similar to obtain.

For system identification, the parameters p + 1 = q = 40 were chosen, when computing the subspace matrices $\mathbf{H}^{(j)}$, $j = 1,\ldots,4$, in (7). When computing the correlations R_i, the reference-based SSI variant (Peeters and De Roeck 1999) was chosen, where the R_i are the correlations between the output of all the sensors on the left side and only the reference sensor on the right side. From the merged observability matrix $\mathbf{O}^{(all)}$ obtained from the merging algorithm described in Section 2.3, system identification was performed at model orders $n = 1,\dots,40$ by choosing the appropriate columns of $\mathbf{O}^{(all)}$.

The resulting stabilization diagram of the natural frequencies with their standard deviations is presented in Figure 2. In the stabilization diagram a threshold on the relative standard deviation (standard deviation of the value divided by this value) was put to delete frequencies with a high uncertainty, as they indicate spurious modes. The obtained modal parameters together with their relative standard deviations at model order 40 are presented in Table 1. The respective mode shape estimates are presented in Figure 3.

Table 1: Identified frequencies (f) and damping ratios (d) together with their relative standard deviations.

	f Hz	$\sigma_f / f \cdot 100$ %	d %	$\sigma_d / d \cdot 100$ %
Mode 1	358.1	0.40	2.1	9.0
Mode 2	551.9	0.15	2.6	6.8
Mode 3	787.5	0.36	3.6	16
Mode 4	923.4	0.21	2.4	8.5
Mode 5	1096	0.09	2.2	4.6
Mode 6	1262	0.86	3.5	20
Mode 7	1508	0.11	2.5	3.6
Mode 8	1855	0.43	2.7	27
Mode 9	1928	0.45	2.7	31

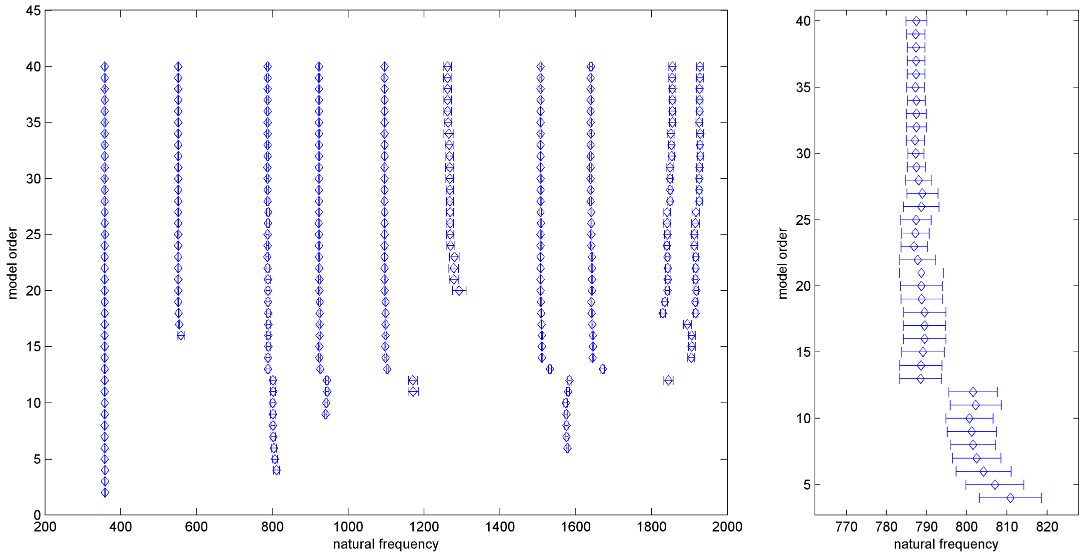

Figure 2: Stabilization diagram with standard deviations of frequencies and zoom on mode 3.

5 CONCLUSIONS

In this paper, we presented the uncertainty quantification of estimated modal parameters from multi-setup measurments, where the modal parameters are estimated in a global system identification step after normalizing and merging the data from the different setups, and not on each setup separately. System identification results of a composite panel were shown and their uncertainties computed.

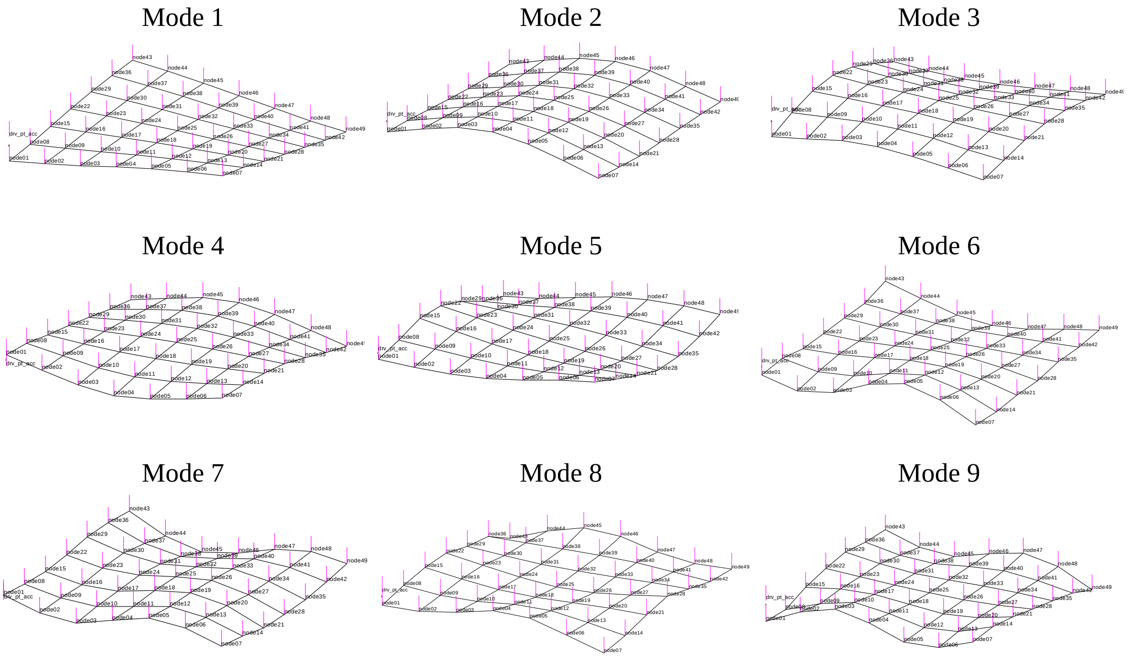

Figure 3: Mode shapes obtained from multi-setup system identification (Luczak et al. 2010).

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