Operational Modal Analysis using a Fast Stochastic Subspace Identification Method

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ABSTRACT

Stochastic subspace identification methods are an efficient tool for system identification of mechanical systems in Operational Modal Analysis, where modal parameters (natural frequencies, damping ratios, mode shapes) are estimated from measured ambient vibration data of a structure. System identification is usually done for many successive model orders, as the true system order is unknown. Then, identification results at different model orders are compared to distinguish true structural modes from spurious modes in so-called stabilization diagrams. These diagrams are a popular GUI-assisted way to select the identified system model, as the true structural modes tend to be stable for successive model orders, fulfilling certain stabilization criteria that are evaluated in an automated procedure. In Operational Modal Analysis of large structures the number modes of interest as well as the number of used sensors can be very large, thus leading to high model orders that have to be considered for system identification. This also means a big computational burden. Recently, an efficient approach to estimate system matrices at multiple model orders in Stochastic Subspace Identification was proposed. In this paper it is shown how this new "Fast SSI" improves the computation of the stabilization diagrams, leading to much faster system identification results for large systems. The Fast SSI is applied to the system identification of some relevant large scale industrial examples.

1 Introduction

Subspace-based system identification methods have been proven efficient for the identification of linear time-invariant systems (LTI), fitting a linear model to input/output or output-only measurements taken from a system. An overview of subspace methods can be found in [1], [2], [3], [4]. During the last decade, subspace methods found a special interest in mechanical, civil and aeronautical engineering for the identification of *modal parameters* (natural frequencies, damping ratios, mode shapes) of vibrating structures, as they are computationally efficient methods and can deal with realistic excitation assumptions.

In an Operational Modal Analysis (OMA) context, the number of sensors can be large (up to hundreds or thousands in the future), as well as the number of modes to be identified. In order to retrieve the desired large number of modes, an even larger model order must be assumed while performing identification. This over-specification of the model order and effects due to measurements under operational conditions, such as finite number of data samples, measurement noises, non-stationary excitations or nonlinear structure, cause a number of spurious modes to appear in the identified models. Based on the observation that physical modes remain quite constant when estimated at different over-specified model orders, while spurious modes vary, they can be distinguished using the well-known *stabilization diagrams* [3]. There, the physical modes are selected from system identification results at multiple model orders in a GUI-assisted way. As system identification is done at an over-specified model order and repeated while truncating at multiple model orders, the computational burden for this procedure is significant especially for large model orders.

Recently, the authors proposed a fast computation scheme for Stochastic Subspace Identification at multiple model orders in [5], [6]. With these “Fast SSI” algorithms, stabilization diagrams can be computed very fast – especially for structures equipped with many sensors and at high model orders. In this paper, their efficiency is demonstrated on several large scale structures.

2 Stochastic Subspace Identification (SSI)

Stochastic Subspace Identification methods are the state of the art methods for modal parameter estimation. They provide unbiased and consistent estimates, even under non-stationary excitation [1], [4]. In this section, an overview of the identification algorithm is given.

2.1 Models and Parameters

The behavior of a mechanical system is assumed to be described by a stationary linear dynamical system

$$M\ddot{Z}(t) + C\dot{Z}(t) + KZ(t) = v(t), \quad Y(t) = LZ(t)\,, \tag{1}$$

where t denotes continuous time, M, C and K are the mass, damping and stiffness matrices, high-dimensional vector Z collects the displacements of the degrees of freedom of the structure, the non-measured external force v modeled as non-stationary Gaussian white noise, the measurements are collected in the vector Y and matrix L indicates the sensor locations. Let m be the number of degrees of freedom of system (1), such that M, C and K are of dimension m x m, and let r be the number of sensors, such that Y is of dimension r.

The eigenstructure of (1) with the modes μ and mode shapes φ_μ is a solution of

$$\det(\mu^2 M + \mu C + K) = 0, \quad (\mu^2 M + \mu C + K)\phi_\mu = 0, \quad \varphi_\mu = L\phi_\mu\,. \tag{2}$$

Sampling model (1) at some rate $1/\tau$ yields a discrete model in state-space form

$$X_{k+1} = FX_k + V_{k+1}, \quad Y_k = HX_k\,, \tag{3}$$

where the state transition matrix F is of dimension n x n with the model order $n = 2m$, and the observation matrix H is of dimension r x n. The eigenstructure of system (3) is given by

$$\det(F - \lambda I) = 0, \quad (F - \lambda I)\phi_\lambda = 0, \quad \varphi_\lambda = H\phi_\lambda\,. \tag{4}$$

Then, the eigenstructure of the continuous system (1) is related to the eigenstructure of the discrete system (3) by

$$e^{\tau\mu} = \lambda, \quad \varphi_\mu = \varphi_\lambda\,. \tag{5}$$

The collection of modes and mode shapes $(\lambda, \varphi_\lambda)$ is a canonical parameterization of system (3). From the eigenvalues λ, the natural frequencies f and damping ratios d of the system are directly recovered from $f = a\,/\,(2\pi\tau)$ and $d = 100|b|/(a^2 + b^2)^{1/2}$, where $a = |\arctan \mathrm{Re}(\lambda)/\mathrm{Im}(\lambda)|$ and $b = \ln|\lambda|$.

2.2 Data-driven SSI

To obtain the eigenstructure of system (3) from measurements $(Y_k)_{k=1,\ldots,N+p+q}$, the stochastic subspace identification algorithm is used. In the first step, the so-called subspace matrix $\mathbf{H}$ is built according to a selected subspace algorithm. In the following, the data-driven SSI using the UPC algorithm [2], [3] is described, but also any other SSI algorithm can be used.

In the first step, the parameters p and q are chosen and the block data Hankel matrices

$$\mathbf{Y}^{+} = \frac{1}{\sqrt{N}} \begin{pmatrix} Y_{q+1} & Y_{q+2} & \cdots & Y_{N+q} \\ Y_{q+2} & Y_{q+3} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ Y_{q+p+1} & Y_{q+p+2} & \cdots & Y_{N+p+q} \end{pmatrix}, \quad \mathbf{Y}^{-} = \frac{1}{\sqrt{N}} \begin{pmatrix} Y_{q}^{(\mathrm{ref})} & Y_{q+1}^{(\mathrm{ref})} & \cdots & Y_{N+q-1}^{(\mathrm{ref})} \\ Y_{q-1}^{(\mathrm{ref})} & Y_{q}^{(\mathrm{ref})} & \cdots & Y_{N+q-2}^{(\mathrm{ref})} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1}^{(\mathrm{ref})} & Y_{2}^{(\mathrm{ref})} & \cdots & Y_{N}^{(\mathrm{ref})} \end{pmatrix} \tag{6}$$

are filled, where in matrix $Y_k^{(\mathrm{ref})}$ a possible subset of the output sensors can be used, the so-called reference sensors or projection channels for a more economic identification procedure [3]. Let r_0 be the dimension of $Y_k^{(\mathrm{ref})}$. The parameters p and q in (6) are chosen, such that $\min(pr, qr_0) \geq n$.

Theoretically, the projection $\mathbf{H} = \mathbf{Y}^{+}\, \mathbf{Y}^{-T} (\mathbf{Y}^{-}\, \mathbf{Y}^{-T})^{-1}\, \mathbf{Y}^{-}$ is computed, but in practice only the left matrices of a SVD of $\mathbf{H}$ are needed, which can also be obtained from the thin LQ decomposition

$$\begin{pmatrix} \mathbf{Y}^{-} \\ \mathbf{Y}^{+} \end{pmatrix} = \begin{pmatrix} R_{11} & 0 \\ R_{21} & R_{22} \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}. \tag{7}$$

Then, $\mathbf{H}$ is defined as $\mathbf{H} = R_{21}$.

The matrix $\mathbf{H}$ possesses the factorization property $\mathbf{H} = \mathbf{O}\,\mathbf{X}$ into observability matrix $\mathbf{O}$ and some other matrix $\mathbf{X}$, where $\mathbf{O}$ is obtained from $\mathbf{H}$ by a singular value decomposition (SVD) and truncation at the desired model order n

$$\mathbf{H} = \begin{pmatrix} U_1 & U_0 \end{pmatrix} \begin{pmatrix} \Delta_1 & \\ & \Delta_0 \end{pmatrix} V^T, \quad \mathbf{O} = U_1 \Delta_1^{1/2} \overset{\mathrm{def}}{=} \begin{pmatrix} H \\ HF \\ \vdots \\ HF^{p} \end{pmatrix}. \tag{8}$$

Note that Δ_1 contains n singular values and U_1 contains n columns. From the observability matrix $\mathbf{O}$ the matrices H in the first block row and F from a least squares solution of

$$\overline{\mathbf{O}}\, F = \underline{\mathbf{O}} \quad \text{with} \quad \overline{\mathbf{O}} = \begin{pmatrix} H \\ HF \\ \vdots \\ HF^{p-1} \end{pmatrix}, \; \underline{\mathbf{O}} = \begin{pmatrix} HF \\ HF^2 \\ \vdots \\ HF^{p} \end{pmatrix} \tag{9}$$

are obtained. The eigenstructure $(\lambda, \varphi_\lambda)$ of the system (3) is then obtained from (4).

3 Fast Multi-Order Subspace Identification

3.1 Multi-Order System Identification

In order to compute the modal parameters at multiple system orders for a stabilization diagram, system identification needs to be done at different successive model orders $n = n_j$, $j = 1, \ldots, t$, with

$$1 \leq n_1 < n_2 < \ldots < n_t \leq \min(pr, qr_0) \tag{10}$$

where t is the number of models to be estimated. The choice of the model orders n_j, $j = 1, \ldots, t$, is up to the user and also depends on the problem. For example, $n_j = j + c$ or $n_j = 2j + c$ with some constant c can be chosen. As the eigenvalues of the state transition matrix are pairwise complex conjugate, the latter choice can be made in the sense that two model orders are needed to recover one new mode.

To indicate the model order n_j for the respective matrices in the identification procedure, the subscript j is used. Thus, at each model order n_j the matrices F_j and H_j are identified from the observability matrix $\mathbf{O}_j$. Note that the latter consists of the first n_j columns of $\mathbf{O}_t$ due to (8).

3.2 Solution of the Least-Squares Problem for the State Transition Matrix F

The observation matrix H_j is easily obtained from the first block row (containing r rows) of the observability matrix $\mathbf{O}_j$ at each model order. Obtaining F_j is more complicated, as the least squares problem (8) needs to be solved each time, which writes at model order n_j

$$\overline{\mathbf{O}}_j \, F_j = \underline{\mathbf{O}}_j \, . \tag{11}$$

The standard solution of this equation is obtained by using the pseudoinverse

$$F_j = \left(\overline{\mathbf{O}}_j\right)^+ \underline{\mathbf{O}}_j \, , \tag{12}$$

where $^+$ denotes the Moore-Penrose pseudoinverse. A more efficient and also numerically stable way to solve it [7], uses the thin QR decomposition

$$\overline{\mathbf{O}}_j = Q_j R_j \, . \tag{13}$$

With

$$S_j = Q_j^T \underline{\mathbf{O}}_j \tag{14}$$

the solution of the least squares problem is

$$F_j = R_j^{-1} S_j \, . \tag{15}$$

Note that R_j is upper triangular and both R_j and S_j are of size n_j x n_j.

For the identification procedure, the observability matrix $\mathbf{O}_t$ is first obtained at the maximal desired model order n_t. Then, the steps (13) – (15) (or the step (12)) need to be repeated at the different model orders n_j, $j = 1, \ldots, t$, where $\mathbf{O}_j$ is obtained from the first n_j columns of $\mathbf{O}_t$.

3.3 Fast Multi-Order Computation

Solving the least squares problem for the computation of the state transition matrix at multiple model orders is a big computational burden, especially when using a large number of sensors and high model orders. In [5], [6] a fast algorithm was proposed, which exploits the structure of the least squares problem at multiple model orders, namely that $\mathbf{O}_j$ consists of the first n_j columns of $\mathbf{O}_t$.

This algorithm computes the necessary matrices for solving the least squares problem only once at the maximal desired model order n_t (Equations (13) – (15) with $j = t$), leading to matrices R_t, S_t and F_t. Then, instead of solving the least squares problems at all the orders $n_1, n_2, \ldots, n_{t-1}$, it was shown that the state transition matrices F_j at these lower orders can be computed much more efficiently directly from submatrices of R_t and S_t.

Theorem 1 **([5], [6]).** Let $\mathbf{O}_t$, Q_t, R_t and S_t be given at the maximal desired model order n_t with

$$\overline{\mathbf{O}}_t = Q_t R_t\,, \quad S_t = Q_t^T \underline{\mathbf{O}}_t\,, \quad F_t = R_t^{-1} S_t\,, \tag{16}$$

such that F_t is the least squares solution of

$$\overline{\mathbf{O}}_t \; F_t = \underline{\mathbf{O}}_t\,. \tag{17}$$

Let $j \in \{1, \ldots, t-1\}$, and let R_t and S_t be partitioned into blocks

$$R_t = \begin{pmatrix} R_j^{(11)} & R_j^{(12)} \\ 0 & R_j^{(22)} \end{pmatrix}, \quad S_t = \begin{pmatrix} S_j^{(11)} & S_j^{(12)} \\ S_j^{(21)} & S_j^{(22)} \end{pmatrix}, \tag{18}$$

where $R_j^{(11)}$ and $S_j^{(11)}$ are of size n_j x n_j. Then, the state transition matrix F_j at model order n_j, which is the least squares solution of

$$\overline{\mathbf{O}}_j \; F_j = \underline{\mathbf{O}}_j\,, \tag{19}$$

satisfies

$$F_j = \left(R_j^{(11)}\right)^{-1} S_j^{(11)}\,. \tag{20}$$

Using Theorem 1, steps (13) and (14) for the least squares solution of the state transition matrix F_j are not necessary anymore for $j = 1, \ldots, t-1$. Once the matrices R_t and S_t are obtained, the submatrices $R_j = R_j^{(11)}$ and $S_j = S_j^{(11)}$ are selected at each model order n_j from R_t and S_t, and the state transition matrix is directly obtained in (15), which brings a significant reduction of the computational efforts. From these matrices, the eigenstructure identification is done in (3) and (4) in order to fill a stabilization diagram at the multiple model orders for the modal analysis.

For a more detailed analysis of the described algorithm, as well as for an iterative computation of the system matrices, the interested reader may refer to [5], [6].

4 Applications

To demonstrate the efficiency of the new fast Crystal Clear SSI algorithm, some operational data from a ship were analyzed, see Figure 1.

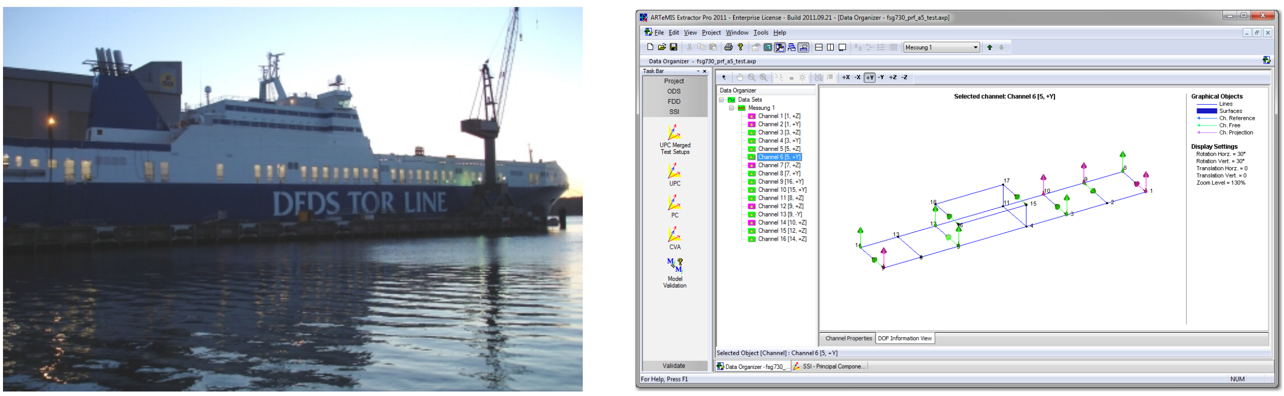

Figure 1: The ship that was measured during operation in 16 channels. Measurements were made by Professor Schlottmann and Dr. Rosenow, Rostock University, Germany. Right picture shows the sensor layout.

The measurements consisted of 16 channels with 691200 samples per channel. This data contains over 40 modes and several harmonics arising from the propellers.

The objective of the example was to test the speed increase using the new algorithm compared to the conventional one. By using the new algorithm, the estimation of the stabilization diagram for model orders ranging from 1 to 300 was finished after only 43 seconds, whereas the conventional algorithm needed 52 minutes. This is equivalent to a speed increase of 73 times in this particular case. Figure 2 presents the stabilization diagram.

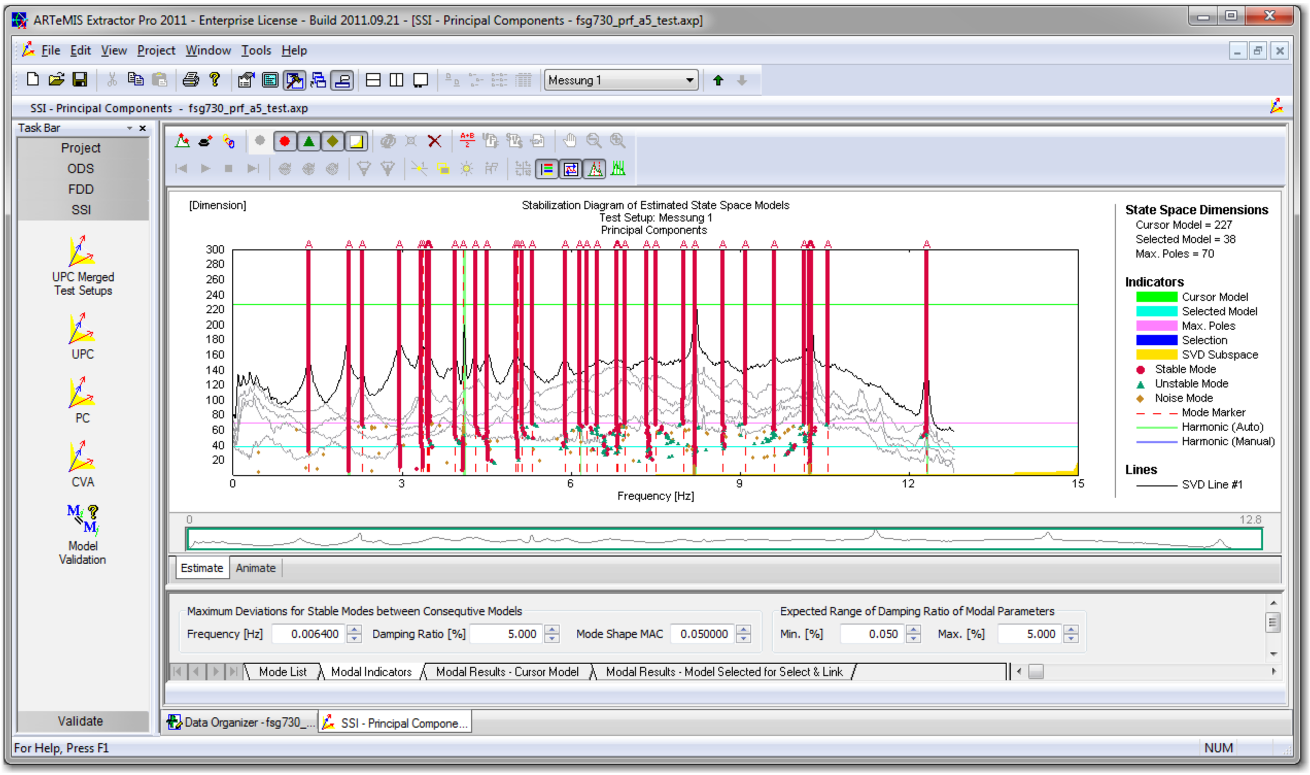

Figure 2: Stabilization diagram obtained in ARTeMIS Extractor Pro.

5 Conclusions

In this paper, a new algorithm was presented that efficiently computes the system matrices at multiple model orders in Stochastic Subspace Identification. The computational complexity for this part of the computation of the stabilization diagram is significantly reduced. Thus, it is a major advancement for subspace-based modal analysis, especially for large structures equipped with many sensors, where high model orders are considered.

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