10 Conclusions

This chapter is divided into three sections. Section 10.1 contains a complete summary of chapters 1 to 9 of the thesis. The second section gives an overall discussion and conclusion on the topics treated in this thesis. Finally, the future perspectives in using multivariate stochastic time domain models for identification of civil engineering structures are given in section 10.3.

10.1 Summary

Chapter 1

Chapter 1 contains the introduction to this thesis. The main purpose of this thesis has been to investigate how to represent the dynamic behaviour of ambient excited civil engineering structures by linear and time-invariant discrete-time stochastic models, represented in time domain. The focus has been put on the use of Auto-Regressive Moving Average Vector (ARMAV) models and equivalent state space realizations.

A secondary purpose has been to make the theory of ARMAV modelling more accessible to civil engineers. Emphasis has been put on relating system identification using the ARMAV model to traditional modal analysis of civil engineering structures. Two experimental cases related to the thesis have been introduced at the end of chapter 1.

Chapter 2

In this chapter the basic theory of linear and time-invariant discrete-time stochastic systems is introduced. The first section relates the ARMAV model not accounting for the presence of noise to a stochastically excited state space system. Noise is then added to the state space system and the corresponding ARMAV model is derived. The last section of this chapter concerns how to represent ARMAV by a particular state space realization. This representation is shown both for the ARMAV model that accounts for the presence of noise and the one that does not. All the derived relations between ARMAV models and equivalent stochastic state space systems are used in the following chapters of the thesis.

Chapter 3

The first section of chapter 3 has concerned the continuous-time mathematical description of structural systems by use of a second-order differential equation system. The concepts of modal and spectrum analysis have also been introduced in this section.

However, since the number of measurement channels in general is different from the number of dynamic modes of a system, it is necessary to generalize the mathematical description of the structural system. Further, to use the ARMAV model as a discrete-time representation of an ambient excited structure, the ambient excitation is assumed to be the output of a linear shaping filter subjected to Gaussian white noise.

Since the ARMAV model assumes a Gaussian white noise, the generalized structural system and the shaping filter have been combined into one continuous-time Gaussian white noise excited mathematical model. This system has been generalized even further to be able to extract other characteristics than the displacements. It has been shown through a modal decomposition of the combined system that it is still possible to extract the modal parameters of the structural system.

Chapter 4

This chapter has considered what happens when the combined continuous-time system is sampled. In this case a discretized approximate model is obtained. The discretization can be performed in a number of ways. However, in the context of ambient excited structures where only the system response is available, the most appropriate model is obtained by the covariance equivalence technique.

This technique require, that the first and second-order moments of the response of the combined continuous-time system must be equal to the first and second-order moments of the response of the discretized model, at all discrete time instances.

In the noise-free case the discretized model of an *n*th order multivariate combined continuous-time system is an ARMAV(n,n-1) model. If measurement noise is present, it is shown that the appropriate model changes to an ARMAV(n,n) model.

Chapter 5

This chapter has considered the estimation of the multivariate ARMAV model, or equivalently the innovation state space representation, using a nonlinear prediction error method (PEM). It has been discussed how to obtain an optimal predictor for a given model structure, and how to assess the performance of this predictor through a scalar valued criterion function. This criterion function has been selected so it provides optimal accuracy of the estimated parameters, under the assumption that the prediction errors form a Gaussian white noise.

The minimization of the criterion function with respect to the adjustable parameters of the model structure and a specific search scheme for iterative improvement of the estimate have been considered.

The statistical properties of the PEM estimator are then investigated. It is shown that if the true system is contained in the model structure and if the prediction errors form a Gaussian white noise, then the estimator will be consistent and asymptotically efficient. Based on these assumptions it is shown how to obtain an estimate of the covariance matrix of the estimated parameters.

Finally, the problem of selecting the correct model structure and how to validate that this model actually will fulfil its purpose has been considered.

Chapter 6

The purpose of this chapter has been to describe modal analysis using discrete-time parametric time domain models in the field of civil engineering. It has been described how to modally decompose a discrete-time parametric model and how to obtain the modal parameters on the basis of this decomposition.

In some applications, it is important to have an idea about the accuracy of the estimated modal parameters. It has therefore been shown how to estimate the standard deviations of the estimated natural eigenfrequencies, damping ratios, and mode shapes.

One of the difficulties in using parametric models for system identification is that the selection of physical and nonphysical modes in general must be performed by the user. In this context, guidelines on how to distinguish between these two types of modes have been given.

As a part of a modal analysis of a structural system a spectrum analysis is often performed. For stochastically excited systems, spectrum analysis is a powerful way to visualize the dynamic properties of the structural system itself and its excitation. It has therefore been shown how to obtain the spectral densities directly from the identified model.

Finally, at some prior state a modal analysis of a structure might have been made using a univariate parametric model. If a more complex analysis using several sensors is desired at a later state, then all modal parameters can be determined by using the prior information about the structural eigenvalues. If this prior information is combined with a covariance estimation approach applied to the new measurements, then the mode shapes of the structural modes can be estimated. This approach, however, should only be applied is the modes are well separated.

Chapter 7

In this chapter the numerical and computational aspects have been considered. It has been described how to prepare the measured data by prefiltering, resampling and detrending to make it suitable for system identification.

It has also been shown how to scale the measured data and balance the estimated models to improve the numerical accuracy. If the measured data contains outliers, the prediction filter that is used should account for this. Otherwise, they can have a significantly bad effect on the performance of the estimator. Further, if the amount of measured data is limited, and if the model order is relatively high then a bad transient behaviour of the prediction filter might occur. If this is the case, it should be eliminated since this also affects the performance of the estimator. It is shown how to account for these problems.

In the chapter the Structural Time Domain Identification (STDI) toolbox for use with MATLAB has been introduced, and its applicability is illustrated by an example based on simulated response of a Gaussian white noise excited second-order system.

Chapter 8

This chapter concerns the first experimental case, which is a simulation study. The purpose has been to assess the statistical properties of the PEM estimate of an ARMAV model. The analysis concerns the influence of the length of measurement record and the noise level on the bias and standard deviation of the modal parameter estimates.

The results indicate that the estimator probably becomes efficient as the record length tends to infinity. This is underlined by an investigation of the bias of the modal parameter estimates. It is concluded that especially the standard deviations of the natural eigenfrequencies and the damping ratios can be quantified accurately even though they tend to be underestimated. The standard deviations of the mode shapes will be more inaccurately estimated and care must be taken in interpreting these.

Chapter 9

This chapter has concerned how system identification using ARMAV models can be used in applications such as Vibration Based Inspection (VBI) as basis for damage detection. By using the ARMAV models, accurate natural eigenfrequency estimates have been obtained and their uncertainties have been estimated. Even though the analysis has only made use of the natural eigenfrequency estimates the ARMAV model has been preferred instead of the ARMA model. The reason is the presence of two sets of closely spaced modes.

It has been possible to detect significant changes of some of the natural eigenfrequency estimates down to a few per cent, due to an introduced damage. By using the estimated standard deviations of the estimated natural eigenfrequencies, the confidence in the significance of these changes has been estimated in a probabilistic sense. The conclusion is that the damage can be detected with a confidence of 95% when it enters its third state.

10.2 General Conclusions

In this thesis system identification of civil engineering structures using parametric stochastic models has been considered. It has been shown that if the structural system can be assumed to be a linear and time-invariant lumped parameter system, and if the excitation can be assumed generated by a linear and time-invariant shaping filter subjected to Gaussian white noise, then the ARMAV model will be an adequate model.

The relation between the combined continuous-time system and the ARMAV model has been derived by assuming a covariance equivalence of the response of the continuous-time system and the ARMAV model for all discrete time steps. If the measured response of a linear and time-invariant structural system is Gaussian distributed and if the excitation is unknown, then the covariance equivalence technique results in a discrete-time model correctly describing the dynamic properties of the structural system as well as the statistical properties of the response.

For an nth order multivariate continuous-time system subjected to Gaussian white noise the covariance equivalent discrete-time model is an ARMAV(n,n-1) model. In other words, a model that is constructed from an nth order auto-regressive matrix polynomial and a moving average matrix polynomial of order n-1. Frequently, this model structure is applied in system identification based on sampled data. However, sampled data are always affected by noise, and it is shown that an ARMAV(n,n) model in general should be used instead of an ARMAV(n,n-1) model when noise is present.

Throughout this thesis the relations between the ARMAV model and the stochastic state space system have been used extensively. The ARMAV representation has been applied to show the effects of covariance equivalence and presence of noise, whereas the state space representation has been applied for the modal decomposition. The switch between the two representations shows that the choice of representation only depends on the actual application.

In this thesis the accuracy of an estimated ARMAV model has been emphasized. For this reason the applied estimation technique is the Prediction Error Method (PEM). The advantage of this estimation technique is that it is asymptotically unbiased and efficient if the prediction errors are Gaussian distributed and if the true system is contained in the estimated model. These asymptotic statistical properties have been verified by a simulation study with special regard to the modal parameter estimates. The disadvantage of using a nonlinear PEM is the computational time needed, compared to algorithms that e.g. rely on the Singular Value Decomposition.

The use of the PEM estimator has made it possible to estimate the standard deviations associated with the modal parameter estimates. However, a simulation study has revealed that only the estimated standard deviations of the natural eigenfrequency estimates are accurate enough for applications such as VBI.

Especially the estimation of the standard deviations of the mode shape coordinates was poor.
In any case, the estimated standard deviations of all the modal parameter estimates can be used as a validation tool. If an identified mode is of physical origin all estimated standard deviations will be small compared to modes of nonphysical origin.

The estimation of the associated uncertainties has been utilized in VBI. By using this additional information about the natural eigenfrequency estimates, it possible to detect whether a change of these is significant with a certain statistical confidence or not.

So to recapitulate the results of this thesis. The ARMAV model estimated using the off-line PEM should be applied in modal analysis if :

- ☞ *The structural system is linear and time-invariant.*
- ☞ *The excitation is unknown.*
- ☞ *The measured response is stationary and can be assumed Gaussian distributed.*
- ☞ *Uncertainty estimates of the modal parameters are needed.*

For a large variety of practical system identification problems in civil engineering, the assumptions concerning linearity, Gaussianity and stationarity are fulfilled. In these cases, system identification using ARMAV models can serve as a reliable and valuable alternative to the traditional non-parametric system identification techniques.

10.3 Future Perspectives

As stated above, the use of ARMAV modelling of the dynamic behaviour of a civil engineering structure and the use of an off-line PEM estimation approach has been based on the following assumptions:

- ☞ *The structural system is linear and time-invariant.*
- ☞ *The measured response is stationary and can be assumed Gaussian distributed.*

Besides these mathematical assumptions there are also some practical considerations in using the off-line PEM approach for estimation of ARMAV models. These assumptions and practical considerations raise the following questions:

- ☞ *What happens if the measured response is non-Gaussian distributed or non-stationary?*
- ☞ *To what extent can the* PEM *estimation approach of* ARMAV *models be used in problems having many observable modes and outputs?*
- ☞ *How can the computational time of the PEM algorithm be reduced?*

It has been emphasized that the system response must be stationary and Gaussian distributed. If these assumptions cannot be fulfilled then the covariance equivalence technique will not result in a proper discrete-time modelling of the system, and the predictor of the PEM algorithm will not be optimal in the least-squares sense. It is therefore important to investigate what happens with the model structure if the Gaussian assumption is violated. Further, it is also necessary to investigate the applicability of the PEM algorithm in these cases. One way to make these investigations is perhaps by means of a simulation study.

If the structural system behaves nonlinear or is time-varying the use of the off-line PEM estimation approach will result in an equivalent linear and time-invariant model. However, the sensitivity of the off-line PEM approach to small nonlinearities or time variant behaviour of the system has not been investigated. The time-varying influence should be analysed, since it can provide information about e.g. when to switch to an on-line estimation algorithm.

The estimation of the ARMAV model using the off-line PEM algorithm is not a problem when the number of observed modes and outputs are below certain limits. These limits depend on the speed and memory of the computer being used. A typical identification problem that can be solved on an average quality computer (Pentium 133 MHz, 35 MB RAM) involves a system having 10 outputs and 10 modes. However, the computational time involved is perhaps an hour or more for such a problem when the number of samples is in the range from 5000 to 10000 per output.

In order to make the off-line PEM algorithm more computationally fast two things must be improved. The initial estimate must be more accurate to minimize the number of iterations needed. Further, the most time consuming part of the algorithm must be altered. During an iteration about 90% of the computational time is spent on the construction of the gradient of the prediction filter. So, if it is possible to optimize this part of the algorithm even further, by e.g. parallel computing, the computational time can be minimized significantly.

In relation to VBI the following questions can be raised:

- ☞ *Can the estimation of the standard deviations of the mode shapes be improved?*
- ☞ *Can the estimated standard deviation of the estimated modal parameters be used as basis to a more reliable statistically based VBI?*

The performed simulation study has revealed that the estimation of the standard deviations of the mode shapes was poor. This can be caused by several things. It will be worthwhile investigating the influence of the linearization of the functional relationship between mode shapes and the model parameters.

Also, the influence of the normalization with respect to one specific mode shape coordinate might influence the results. Because of this normalisation, the standard deviations of the other coordinates of the mode shape are estimated relative to this normalized coordinate, which becomes completely deterministic. The estimates of the standard deviations might be improved if it was possible to “smear” the estimation uncertainties over all coordinates in some way.

In the second experimental case the estimated standard deviations of the natural eigenfrequencies were used to form a probabilistic measure of whether a damage has occurred in the structure or not. It might be possible to improve this technique even more by using several damage indicators and their estimated uncertainties to form a unified damage measure. The result of such a unification would probably be a detection of damage at an earlier state.

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Resume in Danish

Det primære formål med denne Ph.D. afhandling har været at klarlægge anvendelsesmulighederne for systemidentifikation og modalanalyse af bærende konstruktioners dynamiske opførsel ved brug af Auto-Regressive Moving Average Vector (ARMAV) tidsdomæne modeller. Et sekundært formål har været at udbrede kendskabet til disse modeller inden for bygningsingeniørers felt. Som illustration af anvendeligheden af ARMAV modellen er der præsenteret to eksperimentielle eksempler.

2. Introduktion af tidsdiskrete systemer

Som introduktion til emnet gives der en beskrivelse af sammenhængene mellem ARMAV modellen og en stokastisk tilstandsmodel. Disse sammenhænge opskrives både med og uden tilstedeværelsen af støj. Det vises, hvorledes den generelle stokastiske tilstandsformulering kan omskrives til en ARMAV model. Slutteligt vises det, hvorledes ARMAV modellen kan repræsenteres ved en specifik tilstandsmodel. De opstillede sammenhænge benyttes i resten af afhandlingen.

3. Tidskontinuerte systemer

Belastningen, såsom vind og bølger, som en bærende konstruktion udsættes for, er typisk umulig at måle. Hvis en bærende konstruktions dynamiske opførelse kun kan observes gennem dets respons, er det nødvendigt at medtage en beskrivelse af den belastning, som er påført systemet. Ved at antage at den ukendte belastning er generet ved at filtrere en tidskontinuert Gaussisk hvidstøj igennem et lineær og tidsinvariant filter, og ved at kombinere dette filter med den styrende differentialligning for det strukturelle systems dynamiske opførelse, fremkommer et samlet system, som er belastet af en tidskontinuert Gaussisk hvidstøj. Dette system inkluderer således både dynamikken fra det strukturelle system og belastningen.

Ved en modal dekomposition af dette kombinerede system bliver det vist, at modalparametrene, dvs. egenfrekvenser, dæmpningsforhold og egensvingningsformer, for den strukturelle del af systemet er upåvirket af tilstedeværelsen af belastningsfiltret. Det vises også, at med hensyn til bestemmelsen af modalparameterne er det lige meget, om systemets respons er flytninger, hastigheder eller accelerationer.

4. Ækvivalente tidsdiskrete systemer

Den tidsdiskrete udgave af det tidskontinuerte kombinerede system er en ARMAV model. Denne model fremkommer ved at antage at systemresponset fra det tidskontinuerte system skal være kovariansækvivalent med det tidsdiskrete systemrespons fra ARMAV modellen for alle diskrete tidsstep.

Hvis det tidskontinuerte system har polynomieordnen *n* vises det, at den kovariansækvivalente udgave er en ARMAV(*n*,*n*-1) model. Med andre ord, en model med en auto-regressiv polynomieorden *n* og en moving average polynomieorden *n*-1.

Den kovariansækvivalente ARMAV model tager ikke hensyn til den uundgåelige tilstedeværelse af støj i målinger. I dette tilfælde vil den korrekte modelstruktur være en ARMAV(*n*,*n*) model i stedet. Støjen i et system opdeles i processtøj og målestøj. Processtøj beskriver de unøjagtigheder der måtte være mellem modellen og det virkelige system. Målestøjen kan karakteriseres som den støj, der introduceres i forbindelse med dataopsamlingen. På grund af denne opdeling af støjen vises det, at det alene er tilstedeværelsen af målestøj, der afgør, om den korrekte modelstruktur er en ARMAV(*n*,*n*-1) model eller en ARMAV(*n*,*n*).

5. Systemidentifikation ved brug af Prediction Error Metoder

Den fastlagte modelstruktur kan herefter benyttes til systemidentifikation udelukkende på grundlag af målt respons fra en bærende konstruktion. I denne afhandling er estimationen af ARMAV modellen baseret på den såkaldte Prediction Error Method (PEM). Selve optimerings proceduren er ikke lineær og er benævnt Gauss-Newton metoden.

Grunden til valget af denne metode er dens asymptotiske statistiske egenskaber. Såfremt den valgte modelstruktur kan rumme det virkelige system og såfremt predictionsfejlene udgør en Gaussisk hvidstøj, vil denne metode være asymptotisk effektiv. Dette betyder, at metoden giver estimater med minimum usikkerhed og uden systematiske fejl (bias), når antallet af målinger går mod uendeligt.

Til slut er det blevet beskrevet, hvorledes en optimal model udvælges fra en gruppe af estimerede modeller, og om denne models præstation er tilfredsstillende.

6. Modalanalyse af bærende konstruktioner

Sammenhængen mellem den estimerede ARMAV model og modalparametrene for en bærende konstruktion er blevet klarlagt. Yderligere er der givet retningslinjer for hvorledes usikkerheder på de estimerede modalparametre kan estimeres vha. information fra PEM optimeringen. Denne information vises senere at være overordenlig anvendelig.

Den estimerede ARMAV model vil typisk både indeholde fysiske og ikke-fysiske egensvingninger. De fysiske egensvinger hidrører fra den bærende konstruktion, hvorimod de ikke-fysiske hidrører fra støj og den ukendte belastning. Der er derfor udstukket retningslinjer for, hvordan de fysiske egensvingninger kan udvælges.

Som et vigtigt redskab i forbindelse med modalanalyse benyttes spektrumanalyse ofte. Det er derfor vist, hvorledes den spektrale tæthedsfunktion af en estimeret ARMAV model kan beregnes.

7. Numeriske aspekter og implementering

Der er givet retningslinjer for, hvorledes målinger bør signalbehandles inden den egentlige systemidentifikation. Der er også givet retningslinjer for, hvordan den numeriske nøjagtighed af en estimeret model kan forøges. Endeligt er det beskrevet, hvordan de i afhandlingen omtalte teknikker er blevet implementeret i en MATLAB baseret toolbox.

8. Eksperimentelt eksempel nr. 1

Det første af de to eksperimentielle eksempler er et simuleringsstudium. Formålet med dette studium har været at eftervise de asymptotiske statistiske egenskaber ved PEM optimeringsmetoden. Studiet har omhandlet indflydelsen af antallet af målinger og signal-støj forhold på de estimerede modalparametres usikkerheder og bias.

Resultaterne af simuleringsstudiet underbygger de asymptotiske egenskaber af den benyttede metode. Det er desuden konkluderet at især usikkerhederne af de estimerede egenfrekvenser kan estimeres nøjagtigt.

9. Eksperimentelt eksempel nr. 2

Det andet af de to eksperimentielle eksempler viser, hvorledes nøjagtigheden af de estimerede ARMAV modeller kan benyttes i forbindelse med vibrationsbaseret inspektion (VBI) til skadesdetektering.

Der er foretaget målinger på en 20 m høj gittermast over en længere periode. Efter at have foretaget målinger af mastens uskadede tilstand er en revne blevet introduceret i en af de nederste gitterdiagonaler. Tre gange i den resterende del af perioden er revnens dybde blevet forøget.

Detektering af en skade er defineret som en signifikante ændring af en eller flere egenfrekvenser. Ved at bruge ARMAV modeller estimeret ved PEM teknikken har det været muligt at estimere standardafvigelserne på de estimerede egenfrekvenser. Denne ekstra information er blevet udnyttet til formulering af et simplet probabilistisk skadeskriterium.

Ved brug af dette kriterium kan skaden detekteres på et 95% signifikansniveau i det øjeblik, revnens dybde udvides for tredje gang.

10. Konklusioner

Det konkluderes, at systemidentifikation ved brug af ARMAV modeller estimeret vha. PEM teknikken kan benyttes til fulde, såfremt:

- ☞ *Belastningen på den bærende konstruktion er ukendt.*
- ☞ *Modellen for den bærende konstruktion kombineret med belastning er lineær og tidsinvariant.*
- ☞ *Det målte respons er stationært og Gaussisk fordelt.*
- ☞ *Estimater på usikkerhederne af estimerede modalparametre ønskes.*

Appendix A. Stochastic State Space Systems

Since it is assumed that all stochastic variables in this thesis are Gaussian distributed it seems appropiate to review some facts concerning Gaussian stochastic processes.

A.1 Gaussian Stochastic Processes

This section concerns Gaussian stochastic variables and Gaussian stochastic processes. The main results of this section are formulas for transformation of mean and covariance, and for conditional mean and covariance. These formulas will be applied to the stochastic state space systems described in the proceeding sections. A stochastic variable $\boldsymbol{X}$ of dimension n is said to have a Gaussian probability density of mean $\boldsymbol{\mu}$ and covariance $\boldsymbol{\Sigma}$ if

$$p(x) = [(2\pi)^n \det(\boldsymbol{\Sigma})]^{-1/2} \exp[-\tfrac{1}{2}(x-\mu)\boldsymbol{\Sigma}^{-1}(x-\mu)] \quad \text{(A.1)}$$

The following lemmas state some of the important properties of Gaussian stochastic variables.

Lemma A.1 - Gaussian Stochastic Variables

If $\boldsymbol{X}$ is a multivariate Gaussian stochastic variable of mean $\boldsymbol{\mu}_x$ and covariance $\boldsymbol{\Sigma}_x$, and if $\boldsymbol{Y}$ is defined as a linear combination of $\boldsymbol{X}$,

$$\boldsymbol{Y} = \boldsymbol{B}\boldsymbol{X} + \boldsymbol{\nu} \quad \text{(A.2)}$$

then $\boldsymbol{Y}$ has mean and covariance given by

$$\boldsymbol{\mu}_{\boldsymbol{Y}} = \boldsymbol{B}\boldsymbol{\mu}_{\boldsymbol{x}} + \boldsymbol{\nu} \quad \text{(A.3)}$$

$$\boldsymbol{\Sigma}_{\boldsymbol{Y}} = \boldsymbol{B}\boldsymbol{\Sigma}_{\boldsymbol{X}}\boldsymbol{B}^T \quad \text{(A.4)}$$

□

Lemma A.2 - Partitioning of Gaussian Stochastic Variables

If $\boldsymbol{X}$ is a multivariate Gaussian stochastic variable of mean $\boldsymbol{\mu}_x$ and covariance $\boldsymbol{\Sigma}_x$, and if $\boldsymbol{X}$, $\boldsymbol{\mu}_x$, and $\boldsymbol{\Sigma}_x$ are partitioned as

$$\boldsymbol{X} = \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix}, \quad \boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{bmatrix}, \quad \boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{bmatrix} \tag{A.5}$$

then $\boldsymbol{x}_1$ is a Gaussian stochastic variable with mean $\boldsymbol{\mu}_1$, and covariance $\boldsymbol{\Sigma}_{11}$. □

Lemma A.3 - Conditional Distribution

If $\boldsymbol{X}$ is as in (A.5) the conditional distribution for $\boldsymbol{x}_1$ given $\boldsymbol{x}_2$ is Gaussian with mean and covariance given by

$$E[\boldsymbol{x}_1|\boldsymbol{x}_2] = \boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\boldsymbol{x}_2 - \boldsymbol{\mu}_2) \tag{A.6}$$

$$\boldsymbol{\Sigma}_{\boldsymbol{X}_1|\boldsymbol{X}_2} = \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21} \tag{A.7}$$

□

Lemma A.4 - Orthogonal Projection

Let the zero-mean vector $\boldsymbol{\varepsilon}_1$ be defined as

$$\begin{aligned} \boldsymbol{\varepsilon}_1 &= \boldsymbol{x}_1 - E[\boldsymbol{x}_1|\boldsymbol{x}_2] \\ &= \boldsymbol{x}_1 - \hat{\boldsymbol{x}}_1 \end{aligned} \tag{A.8}$$

$\boldsymbol{\varepsilon}_1$ is then the estimation error between $\boldsymbol{x}_1$ and $\hat{\boldsymbol{x}}_1$, and is uncorrelated and independent of the conditional mean $\hat{\boldsymbol{x}}_1$ and the conditioning vector $\boldsymbol{x}_2$. In other words

$$E[\boldsymbol{\varepsilon}_1\boldsymbol{x}_2^T] = \boldsymbol{0}, \quad E[\boldsymbol{\varepsilon}_1\hat{\boldsymbol{x}}_1^T] = \boldsymbol{0} \tag{A.9}$$

Because $\boldsymbol{\varepsilon}_1$ is a zero-mean stochastic variable, it is orthogonal to $\hat{\boldsymbol{x}}_1$, since the two stochastic variables are uncorrelated and one of the stochastic variables has zero mean. From (A.6) the conditional mean of x_1 given x_2 is

$$\hat{\boldsymbol{x}}_1 = E[\boldsymbol{x}_1|\boldsymbol{x}_2] = \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{x}_2 \tag{A.10}$$

In the present context $\hat{\boldsymbol{x}}_1$ is the image of $\boldsymbol{x}_1$ orthogonally projected onto the manifold spanned by $\boldsymbol{x}_2$. (A.9) itself is called an orthogonal projection. □

It is seen that the component of $\boldsymbol{x}_1$ which is uncorrelated with $\boldsymbol{x}_2$ is simply the part of $\boldsymbol{x}_1$ that is orthogonal to the subspace spanned $\boldsymbol{x}_2$.

Lemma A.5 - Properties of Orthogonal Projections

Let $\boldsymbol{x}_3$ denote a third Gaussian distributed stochastic variable with zero mean. Then the conditional expectation of $\boldsymbol{x}_3$ given $\boldsymbol{x}_2$ and $\boldsymbol{\varepsilon}_1$ is linear in the following sense

$$E[\boldsymbol{x}_3|\boldsymbol{x}_2,\ \boldsymbol{\varepsilon}_1] = E[\boldsymbol{x}_3|\boldsymbol{x}_2] + E[\boldsymbol{x}_3|\boldsymbol{\varepsilon}_1] \tag{A.11}$$

The estimate of $\boldsymbol{x}_3$ given $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ is the same as that obtained when $\boldsymbol{x}_1$ is replaced by $\boldsymbol{\varepsilon}_1$. In other words

$$E[\boldsymbol{x}_3|\boldsymbol{x}_2,\ \boldsymbol{x}_1] = E[\boldsymbol{x}_3|\boldsymbol{x}_2] + E[\boldsymbol{x}_3|\boldsymbol{\varepsilon}_1] = E[\boldsymbol{x}_3|\boldsymbol{x}_2,\ \boldsymbol{\varepsilon}_1] \tag{A.12}$$

□

A.2 Covariance Function of Continuous-Time State Space Systems

This section describes how to calculate the covariance functions of a continuous-time stochastic state space system. This state space system is given in the following definition.

Definition A.1 - Continuous-Time State Space System

The continuous-time state space system is defined as

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{F}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \ , \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t) \end{aligned} \tag{A.13}$$

where

- $\boldsymbol{x}(t)$: $n \times 1$ zero-mean state vector.
- $\boldsymbol{u}(t)$: $nu \times 1$ independent Gaussian distributed stochastic process.
- $\boldsymbol{y}(t)$: $ny \times 1$ system output.
- $\boldsymbol{F}$: $n \times n$ state matrix.
- $\boldsymbol{B}$: $n \times nu$ input matrix.
- $\boldsymbol{C}$: $ny \times n$ observation matrix.

$\boldsymbol{u}(t)$ is a Gaussian white noise process with zero-mean and covariance, described by $E[\boldsymbol{u}(t)\boldsymbol{u}^T(s)] = \boldsymbol{W}\delta(t-s)$. In what follows $\boldsymbol{x}_0$ is assumed independent of $\boldsymbol{u}(t)$. □

The covariance function of the system defined above is derived in the following theorem.

Theorem A.1 - Covariance Function of a White Noise Excited Continuous-Time System

The covariance function $\Gamma(\tau)$ of the weakly stationary output vector $\boldsymbol{y}(t)$ of (A.13) at time lag τ is given by

$$\Gamma(\tau) = \boldsymbol{C}\Pi(\tau)\boldsymbol{C}^T \tag{A.14}$$

$\Pi(\tau)$ is the covariance function of the state vector $\boldsymbol{x}(t)$ at lag τ, defined as

$$\Pi(\tau) = e^{\boldsymbol{F}\tau}\Pi(0) \tag{A.15}$$

where covariance matrix $\Pi(0)$ can be obtained as the positive definite solution of the Lyapunov equation

$$\boldsymbol{F}\Pi(0) + \Pi(0)\boldsymbol{F}^T = -\boldsymbol{B}\boldsymbol{W}\boldsymbol{B}^T \tag{A.16}$$

Proof:

Multiplying the state equation in (A.13) by $\boldsymbol{x}^T(s)$ and taking the expectation yield

$$E\left[\dot{\boldsymbol{x}}(t)\boldsymbol{x}^T(s)\right] = \boldsymbol{F}E\left[\boldsymbol{x}(t)\boldsymbol{x}^T(s)\right] + \boldsymbol{B}E\left[\boldsymbol{u}(t)\boldsymbol{x}^T(s)\right] \tag{A.17}$$

This is in fact a first-order differential equation in the stationary covariance $\Pi(t\text{-}s)$. Since $\boldsymbol{u}(t)$ is independent of $\boldsymbol{x}(t)$ and zero-mean the solution to this differential equation is

$$\Pi(t-s) = e^{\boldsymbol{F}(t-s)}\Pi(0) \tag{A.18}$$

which is equal to (A.15) with $\tau = t - s$. Note that

$$\frac{d}{dt}E\left[\boldsymbol{x}(t)\boldsymbol{x}^T(t)\right] = E\left[\dot{\boldsymbol{x}}(t)\boldsymbol{x}^T(t)\right] + E\left[\boldsymbol{x}(t)\dot{\boldsymbol{x}}^T(t)\right] \tag{A.19}$$

Replace $\dot{\boldsymbol{x}}(t)$ in (A.19) by the state equation to yield

$$\begin{aligned}\frac{d}{dt}E\left[\boldsymbol{x}(t)\boldsymbol{x}^T(t)\right] = {} & \boldsymbol{F}E\left[\boldsymbol{x}(t)\boldsymbol{x}^T(t)\right] + E\left[\boldsymbol{x}(t)\boldsymbol{x}^T(t)\right]\boldsymbol{F}^T + \\ & \boldsymbol{B}E\left[\boldsymbol{u}(t)\boldsymbol{x}^T(t)\right] + E\left[\boldsymbol{x}(t)\boldsymbol{u}^T(t)\right]\boldsymbol{B}^T\end{aligned} \tag{A.20}$$

A solution of the state space equation is given by

$$\boldsymbol{x}(t) = \boldsymbol{x}_0 + \int_0^t \left(\boldsymbol{F}\boldsymbol{x}(\tau) + \boldsymbol{B}\boldsymbol{u}(\tau)\right)d\tau \tag{A.21}$$

The expectation $E[\boldsymbol{x}(t)\boldsymbol{u}^T(t)]$ in (A.20) can then be expressed as

$$\begin{aligned}E\left[\boldsymbol{x}(t)\boldsymbol{u}^T(t)\right] &= E\left[\boldsymbol{x}_0\boldsymbol{u}^T(t)\right] + \\ &\quad \int_0^t \left(\left[\boldsymbol{F}\boldsymbol{x}(\tau)\boldsymbol{u}^T(t)\right] + \boldsymbol{B}E\left[\boldsymbol{u}(\tau)\boldsymbol{u}^T(t)\right]\right)d\tau \\ &= \frac{1}{2}\boldsymbol{B}\boldsymbol{W}\end{aligned} \tag{A.22}$$

where the integration has been performed over a half Dirac delta spike. Inserting this into (A.20) yields

$$\begin{aligned}\frac{d}{dt}E\left[\boldsymbol{x}(t)\boldsymbol{x}^T(t)\right] = {} & \boldsymbol{F}E\left[\boldsymbol{x}(t)\boldsymbol{x}^T(t)\right] + E\left[\boldsymbol{x}(t)\boldsymbol{x}^T(t)\right]\boldsymbol{F}^T + \\ & \boldsymbol{B}\boldsymbol{W}\boldsymbol{B}^T\end{aligned} \tag{A.23}$$

When t approaches infinity $E[\boldsymbol{x}(t)\boldsymbol{x}^T(t)]$ approaches the steady-state covariance $\mathbf{\Pi}(0)$ and the derivative of $E[\boldsymbol{x}(t)\boldsymbol{x}^T(t)]$ approaches zero. Combining these results with (A.23) results in the Lyapunov equation in . The positive definite solution of this equation will as such be the zero-lag weakly stationary covariance matrix. □

A.3 Covariance Function of Discrete-Time State Space Systems

This section describes how to calculate the covariance functions of a discrete-time stochastic innovation state space system defined as

$$\begin{aligned} \hat{\boldsymbol{x}}(t_{k+1}|t_k) &= \boldsymbol{A}\hat{\boldsymbol{x}}(t_k|t_{k-1}) + \boldsymbol{K}\boldsymbol{e}(t_k) , \quad \boldsymbol{e}(t_k) \in NID(\boldsymbol{0},\boldsymbol{\Lambda}) \\ \boldsymbol{y}(t_k) &= \boldsymbol{C}\hat{\boldsymbol{x}}(t_k|t_{k-1}) + \boldsymbol{e}(t_k) \end{aligned} \tag{A.24}$$

The solution of this system can be obtained by recursive substitutions of the state equation itself to yield

$$\begin{aligned} \hat{\boldsymbol{x}}(t_k|t_{k-1}) &= \boldsymbol{A}^s\hat{\boldsymbol{x}}(t_{k-s}|t_{k-s-1}) + \sum_{j=1}^{s}\boldsymbol{A}^{s-j}\boldsymbol{K}\boldsymbol{e}(t_{k+j-s-1}) \\ \boldsymbol{y}(t_k) &= \boldsymbol{C}\hat{\boldsymbol{x}}(t_k|t_{k-1}) + \boldsymbol{e}(t_k) \\ \boldsymbol{e}(t_k) &\in NID(\boldsymbol{0},\boldsymbol{\Lambda}) , \quad \hat{\boldsymbol{x}}(t_{k-s}|t_{k-s-1}) \equiv \hat{\boldsymbol{x}}_0 \end{aligned} \tag{A.25}$$

The covariance function of the system defined above is derived in the following theorem.

Theorem A.2 - Covariance Function of the Output of the Innovation State Space System

The covariance function $\boldsymbol{\Sigma}(s)$ of the stationary output vector $\boldsymbol{y}(t_k)$ of (A.24) at time lag s is given by

$$\boldsymbol{\Sigma}(s) = \begin{cases} \boldsymbol{C}\boldsymbol{\Pi}(0)\boldsymbol{C}^T + \boldsymbol{\Lambda} , & s = 0 \\ \boldsymbol{C}\boldsymbol{A}^{s-1}\boldsymbol{M} , & s > 0 \end{cases} \tag{A.26}$$

with $\boldsymbol{M}$ being the cross-covariance matrix between $\hat{\boldsymbol{x}}(t_{k+1}|t_k)$ and $\boldsymbol{y}(t_k)$

$$\boldsymbol{M} = \boldsymbol{A}\boldsymbol{\Pi}(0)\boldsymbol{C}^T + \boldsymbol{K}\boldsymbol{\Lambda} \tag{A.27}$$

and the covariance matrix $\boldsymbol{\Pi}(0)$ being a positive definite solution of the Lyapunov equation

$$\boldsymbol{\Pi}(0) = \boldsymbol{A}\boldsymbol{\Pi}(0)\boldsymbol{A}^T + \boldsymbol{K}\boldsymbol{\Lambda}\boldsymbol{K}^T \tag{A.28}$$

Proof :

Define $\boldsymbol{\Pi}(s)$ as the covariance at time lag s of the state vector $\hat{\boldsymbol{x}}(t_k|t_{k-1})$. The zero-lag covariance $\boldsymbol{\Sigma}(0)$ is obtained directly from the observation equation as

$$\begin{aligned} \boldsymbol{\Sigma}(0) &= E\Big[\boldsymbol{y}(t_k)\boldsymbol{y}^T(t_k)\Big] \\ &= E\Big[\Big(\boldsymbol{C}\hat{\boldsymbol{x}}(t_k|t_{k-1}) + \boldsymbol{e}(t_k)\Big)\Big(\boldsymbol{C}\hat{\boldsymbol{x}}(t_k|t_{k-1}) + \boldsymbol{e}(t_k)\Big)^T\Big] \\ &= \boldsymbol{C}\boldsymbol{\Pi}(0)\boldsymbol{C}^T + \boldsymbol{\Lambda} \end{aligned} \tag{A.29}$$

The covariance $\boldsymbol{\Sigma}(s)$ at time lag s, for $s > 0$, is given by

$$\begin{aligned} \boldsymbol{\Sigma}(s) &= E\Big[\boldsymbol{y}(t_k)\boldsymbol{y}^T(t_{k-s})\Big] \\ &= E\Big[\Big(\boldsymbol{C}\hat{\boldsymbol{x}}(t_k|t_{k-1}) + \boldsymbol{e}(t_k)\Big)\Big(\boldsymbol{C}\hat{\boldsymbol{x}}(t_{k-s}|t_{k-s-1}) + \boldsymbol{e}(t_{k-s})\Big)^T\Big] \\ &= \boldsymbol{C}E\Big[\hat{\boldsymbol{x}}(t_k|t_{k-1})\hat{\boldsymbol{x}}^T(t_{k-s}|t_{k-s-1})\Big]\boldsymbol{C}^T + \boldsymbol{C}E\Big[\hat{\boldsymbol{x}}(t_k|t_{k-1})\boldsymbol{e}^T(t_{k-s})\Big] \end{aligned} \tag{A.30}$$

It has implicitly been used that $E\Big[\boldsymbol{e}(t_k)\hat{\boldsymbol{x}}^T(t_{k-s}|t_{k-s-1})\Big] = \boldsymbol{0}$ and $E\Big[\boldsymbol{e}(t_k)\boldsymbol{e}^T(t_{k-s})\Big] = \boldsymbol{0}$, for $s > 0$. Observe the following relationships

$$\begin{aligned} \boldsymbol{\Pi}(s) &= E\Big[\hat{\boldsymbol{x}}(t_k|t_{k-1})\hat{\boldsymbol{x}}^T(t_{k-s}|t_{k-s-1})\Big] \\ &= E\left[\left(\boldsymbol{A}^s\hat{\boldsymbol{x}}(t_{k-s}|t_{k-s-1}) + \sum_{j=1}^{s}\boldsymbol{A}^{s-j}\boldsymbol{K}\boldsymbol{e}(t_{k+j-s-1})\right)\hat{\boldsymbol{x}}^T(t_{k-s}|t_{k-s-1})\right] \\ &= \boldsymbol{A}^sE\Big[\hat{\boldsymbol{x}}(t_{k-s}|t_{k-s-1}))\hat{\boldsymbol{x}}^T(t_{k-s}|t_{k-s-1})\Big] \\ &= \boldsymbol{A}^s\boldsymbol{\Pi}(0) \end{aligned} \tag{A.31}$$

$$\begin{aligned} E\Big[\hat{\boldsymbol{x}}(t_k|t_{k-1})\boldsymbol{e}^T(t_{k-s})\Big] &= E\left[\left(\boldsymbol{A}^s\hat{\boldsymbol{x}}(t_{k-s}|t_{k-s-1}) + \sum_{j=1}^{s}\boldsymbol{A}^{s-j}\boldsymbol{K}\boldsymbol{e}(t_{k+j-s-1})\right)\boldsymbol{e}^T(t_{k-s})\right] \\ &= \boldsymbol{A}^{s-1}\boldsymbol{K}E\Big[\boldsymbol{e}(t_{k-s})\boldsymbol{e}^T(t_{k-s})\Big] \\ &= \boldsymbol{A}^{s-1}\boldsymbol{K}\boldsymbol{\Lambda} \end{aligned} \tag{A.32}$$

Inserting (A.31) and (A.32) into (A.30) yields (A.26) with $\boldsymbol{M}$ defined in (A.27). The zero-lag covariance matrix $\boldsymbol{\Pi}(0)$ can be found from the following use of the state equation (A.24)

$$
\begin{aligned}
\Pi(0) &= E\left[\hat{\boldsymbol{x}}(t_{k+1}|t_k)\hat{\boldsymbol{x}}^T(t_{k+1}|t_k)\right] \\
&= E\left[\left(\boldsymbol{A}\hat{\boldsymbol{x}}(t_k|t_{k-1}) + \boldsymbol{K}\boldsymbol{e}(t_k)\right)\left(\boldsymbol{A}\hat{\boldsymbol{x}}(t_k|t_{k-1}) + \boldsymbol{K}\boldsymbol{e}(t_k)\right)^T\right] \\
&= \boldsymbol{A}E\left[\hat{\boldsymbol{x}}(t_k|t_{k-1})\hat{\boldsymbol{x}}^T(t_k|t_{k-1})\right]\boldsymbol{A}^T + \boldsymbol{K}E\left[\boldsymbol{e}(t_k)\boldsymbol{e}^T(t_k)\right]\boldsymbol{K}^T \\
&= \boldsymbol{A}\Pi(0)\boldsymbol{A}^T + \boldsymbol{K}\Lambda\boldsymbol{K}^T
\end{aligned}
\tag{A.33}
$$

Where it has been implicitly have been used that $E\left[\boldsymbol{A}\hat{\boldsymbol{x}}(t_k|t_{k-1})\boldsymbol{e}^T(t_k)\right] = \boldsymbol{0}$ and $E\left[\boldsymbol{e}(t_k)\hat{\boldsymbol{x}}^T(t_k|t_{k-1})\boldsymbol{A}^T\right] = \boldsymbol{0}$. □

Appendix B. Solving a Matrix Equation of Second Order

This appendix describes how to solve the matrix polynomial (4.34) in theorem 4.2. Based on covariance information of the continuous-time structural system, and the $n \times n$ auto-regressive coefficient matrices $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$, two $n \times n$ matrices $\boldsymbol{K}_0$ and $\boldsymbol{K}_1$ are derived. By combining these with $\boldsymbol{A}_1$, the matrix polynomial in (4.33) appears as

$$\boldsymbol{B}_2^2 - \boldsymbol{B}_2\left(\boldsymbol{K}_0 + \boldsymbol{K}_1\boldsymbol{A}_1^T\right)\boldsymbol{K}_1^{-T} + \boldsymbol{K}_1\boldsymbol{K}_1^{-T} = \boldsymbol{0} \qquad \text{(A.34)}$$

where $\boldsymbol{B}_2$ is the unknown $n \times n$ moving average coefficient matrix. The special feature of this polynomial is that $\boldsymbol{B}_2$ is a square matrix and not a vector. This implies that there will be several independent solutions to the equation. Instead of solving (B.1), it is more convenient to solve the transpose of it. Introduce two square matrices $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ as

$$\boldsymbol{D}_1 = -\boldsymbol{K}_1^{-1}\left(\boldsymbol{K}_0 + \boldsymbol{K}_1\boldsymbol{A}_1^T\right)^T, \quad \boldsymbol{D}_2 = \boldsymbol{K}_1^{-1}\boldsymbol{K}_1^T \qquad \text{(A.35)}$$

and $\boldsymbol{X} = \boldsymbol{B}_2^T$. (B.1) can then be expressed as

$$\boldsymbol{X}^2 + \boldsymbol{D}_1\boldsymbol{X} + \boldsymbol{D}_2 = \boldsymbol{0} \qquad \text{(A.36)}$$

which can be solved for unknown $\boldsymbol{X}$. The following solution does not assume that the matrices $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ are symmetrical, since this will prevent the use of non-proportional damping.

Assume that $\boldsymbol{X}$ has n distinct eigenvalues λ_i and can be represented by the following similarity

$$\begin{aligned}\boldsymbol{X} &= \begin{bmatrix}\phi_1 & \phi_2 & . & . & \phi_n\end{bmatrix}\begin{bmatrix}\lambda_1 & 0 & . & . & 0\\ 0 & \lambda_2 & . & . & 0\\ . & . & . & . & .\\ . & . & . & . & .\\ 0 & 0 & . & . & \lambda_n\end{bmatrix}\begin{bmatrix}\phi_1 & \phi_2 & . & . & \phi_n\end{bmatrix}^{-1} \\ &= \boldsymbol{\Phi}\boldsymbol{\Lambda}\boldsymbol{\Phi}^{-1}\end{aligned} \qquad \text{(A.37)}$$

where ϕ_i are the $n \times 1$ corresponding eigenvectors. If this assumption is fulfilled (B.3) can be written as

$$\boldsymbol{\Phi}\boldsymbol{\Lambda}^2\boldsymbol{\Phi}^{-1} + \boldsymbol{D}_1\boldsymbol{\Phi}\boldsymbol{\Lambda}\boldsymbol{\Phi}^{-1} + \boldsymbol{D}_2 = \mathbf{0} \tag{A.38}$$

or equivalently as

$$\boldsymbol{\Phi}\boldsymbol{\Lambda}^2 + \boldsymbol{D}_1\boldsymbol{\Phi}\boldsymbol{\Lambda} + \boldsymbol{D}_2\boldsymbol{\Phi} = \mathbf{0} \tag{A.39}$$

which by definition is n second-order eigenvalue problems. Since there are $2n$ solutions to (B.6), it is necessary to choose n of these in order to construct $\boldsymbol{\Phi}$ and $\boldsymbol{\lambda}$. This implies that there are $\binom{2n}{n}$ solutions for $\boldsymbol{X}$. Without loss of generality each of these $2n$ eigenvalue problems can be linearized, see Gohberg et al. [14], in the following way

$$\begin{bmatrix} \phi_i \\ \phi_i\lambda_i \end{bmatrix}\lambda_i - \begin{bmatrix} \mathbf{0} & \boldsymbol{I} \\ -\boldsymbol{D}_2 & -\boldsymbol{D}_1 \end{bmatrix}\begin{bmatrix} \phi_i \\ \phi_i\lambda_i \end{bmatrix} = \mathbf{0} \ , \quad i = 1,\ 2,\ \ldots\ ,\ 2n \tag{A.40}$$

which is a standard first-order eigenvalue problem.

To summarise the solution of $\boldsymbol{B}_2$:

☞ *Start by taking the transpose of the polynomial* (B.1)*, and calculate* $\boldsymbol{D}_1$ *and* $\boldsymbol{D}_2$ *using* (B.2).

☞ *Then solve n of the* $2n$ *eigenvalue problems in* (B.7).

☞ *Construct the eigenvector matrix* $\boldsymbol{\Phi}$ *and the diagonal matrix* $\boldsymbol{\lambda}$ *of the corresponding eigenvalues*.

☞ *The chosen solution is then given by the transpose of the similarity transformation in* (B.4).

Be aware that the eigenvalues and corresponding eigenvectors might come in complex conjugated pairs. If a complex eigenvalue and its associated complex eigenvector are used in $\boldsymbol{\lambda}$ and $\boldsymbol{\Phi}$, then their complex conjugates must be included as well.