3 Continuous-Time Structural Systems

Structures can be regarded as distributed parameter systems characterized by the distribution of the mass, damping and stiffness properties. However, parameter identification of such systems is in general not easy. Thus, with a few exceptions, in most of the literature on testing of structures, the data are analysed based on the assumption that the system is described by one or more linear ordinary differential equations. Because of their simplicity the linear time-invariant lumped parameter models are the most widely used models in structural identification. More complex models such as the linear continuous parameter models and nonlinear models are used only when the lumped-parameter model cannot be used to provide an adequate representation of the structural behaviour. In general, system identification concerns the determination of modal parameters. If the excitation of a structure e.g. is the wind, then the only available information about the dynamic behaviour of a structure is the measured vibrations of it. As will be shown later, system identification using ARMAV models is in these situations capable of providing good estimates of the modal parameters of the structure. However, it is necessary to obtain some kind of understanding of how the discrete-time ARMAV model relates to the modal parameters, i.e. how it relates to the continuous-time lumped parameter model. It is the purpose of this and the following two chapters to provide this understanding. This chapter is restricted to the continuous-time modelling of civil engineering structures.

Section 3.1 concerns the modelling of a structural system using the second-order lumped parameter model and how the modal parameters of this model are defined. This model is then generalized in section 3.2, in order to cover situations where the number of observed masses of the model are different from the total number of modes in the system. It is also shown how the ambient excitation can be modelled. In section 3.3, the model that describes the excitation is combined with the generalized lumped parameter model of the structural system. Two examples will be given as an illustration. Finally, in section 3.4, the modal decomposition of the combined system is investigated. The results are illustrated by an example.

3.1 Modelling of Second-Order Structural Systems

Due to the complexity of structures, parameter estimation is usually simplified by certain assumptions about the structures. Such simplifications can e.g. be that they behave linearly, that they are time-invariant, and that they can be represented by a mass-spring-dashpot model. Since a structure is a continuous system with distributed mass, such a model should in principle have an infinite number of degrees of freedom. However, since it is seldom more than a few dynamic modes that are of interest, it is sufficient to construct a reduced model, capable of describing the behaviour of the dynamic modes of interest in terms of modal parameters. Such a model is termed a lumped parameter model.

3.1.1 Constructing a Mathematical Model

Experience has led to the following mathematical mass-spring-dashpot lumped parameter model for a structure subjected to external loading, see e.g. Kozin et al. [68] and Pi et al. [92]

$$\boldsymbol{M}\ddot{\boldsymbol{z}}(t) + \boldsymbol{C}\dot{\boldsymbol{z}}(t) + \boldsymbol{K}\boldsymbol{z}(t) = \boldsymbol{f}(t) \tag{3.1}$$

$\boldsymbol{M}$, $\boldsymbol{C}$ and $\boldsymbol{K}$ are the mass, damping and stiffness matrices all of dimensions $p \times p$. $\boldsymbol{z}(t)$ and $\boldsymbol{f}(t)$ are the $p \times 1$ displacement and force vectors at the mass points, respectively. The differential equation represent a force equilibrium. The forces of inertia $\boldsymbol{M}\ddot{\boldsymbol{z}}(t)$ are balanced by a set of linear-elastic restoring forces $\boldsymbol{K}\boldsymbol{z}(t)$, viscous damping $\boldsymbol{C}\dot{\boldsymbol{z}}(t)$ and the external forces $\boldsymbol{f}(t)$. This reduced model is capable of describing the motion of p fictitious mass points of the structure. The mass points are fictitious since they represent a discretization of the distributed mass of the structure. It is thus assumed that the distributed forces of inertia of the structure can be discretized into p degrees-of-freedom (DOF). Since the motion of the system is observed at the mass points, $\boldsymbol{M}$ will be diagonal, and due to the Maxwell theorem the stiffness matrix will be symmetric and positive definite. It is the usual assumption in linear vibration theory that the damping matrix is also symmetric, since any non-symmetric part does not dissipate energy. Assuming a linear structure means that the response of the structure, to any combination of forces simultaneously applied, is the sum of individual responses to each of the forces acting alone. This is a good assumption for a variety of structures. The time-invariant assumption implies that the parameters in the model are constants.

3.1.2 Modal Analysis

One of the most important applications of system identification is that estimated models can serve as a basis for a modal analysis of a structure. In this section, it is shown how to obtain the modal parameters from the second-order structural system (3.1). The modal parameters of the jth mode are the modal frequency, the modal damping, the modal vector and the modal scaling, and below is listed how these four modal parameters can be represented, see e.g. Hoen [38].

Modal frequency:

☞ *Eigenvalue,* λ_j.

☞ *Angular eigenfrequencies,* ω_j.

☞ *Natural eigenfrequencies,* f_j.

Modal damping:

☞ *Damping ratios,* ζ_j.

Modal vector:

☞ *Eigenvectors,* $\boldsymbol{\psi}_j$.

☞ *Mode shapes,* $\boldsymbol{\Phi}_j$.

Modal scaling:

☞ *Modal masses, m_j.*

☞ *Residues, $\boldsymbol{R}_j$.*

Now, assume that the structural system is given a unit impulse and then left on its own. The vibrations of the system will then solely be dependent upon the dynamic characteristics of the structure, for which reason they are called free vibrations. Since the vibrations are viscously damped, it is in general necessary to consider a complex eigenvalue problem to determine the modal parameters. The solution of this eigenvalue problem requires the construction of a $2p \times 2p$ matrix, and correspondingly a $2p$ order response vector on which the matrix operates. This vector is also called the state vector of the system (3.1) and is defined in terms of the displacements and velocities of the system

$$\boldsymbol{x}(t) = \begin{bmatrix} \boldsymbol{z}(t) \\ \dot{\boldsymbol{z}}(t) \end{bmatrix} \tag{3.2}$$

This state vector reduces the second-order differential equation system (3.1) to the following first-order differential equation system

$$\boldsymbol{A}\dot{\boldsymbol{x}}(t) + \boldsymbol{B}\boldsymbol{x}(t) = \boldsymbol{u}(t)$$

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{C} & \boldsymbol{M} \\ \boldsymbol{M} & \boldsymbol{0} \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{M} \end{bmatrix}, \quad \boldsymbol{u}(t) = \begin{bmatrix} \boldsymbol{f}(t) \\ \boldsymbol{0} \end{bmatrix} \tag{3.3}$$

The first p rows of (3.3) are the same as the original equation of motion (3.1), whereas the remaining rows are the identities $\boldsymbol{M}\dot{\boldsymbol{x}}(t) - \boldsymbol{M}\dot{\boldsymbol{x}}(t) = \boldsymbol{0}$. The free vibrations of the system (3.3) are then given by

$$\boldsymbol{A}\dot{\boldsymbol{x}}(t) + \boldsymbol{B}\boldsymbol{x}(t) = \boldsymbol{0} \tag{3.4}$$

and the solution form is assumed as

$$\boldsymbol{x}(t) = \boldsymbol{\psi} e^{\lambda t} \tag{3.5}$$

where $\boldsymbol{\psi}$ is a complex vector of dimension $2p \times 1$ and λ is a complex constant. Insertion of (3.5) into (3.4) shows that (3.5) is a solution if and only if $\boldsymbol{\psi}$ is a solution to the first-order eigenvalue problem

$$(\lambda\boldsymbol{A} + \boldsymbol{B})\boldsymbol{\psi} = \boldsymbol{0} \tag{3.6}$$

which leads to the following characteristic polynomial

$$det(\lambda\boldsymbol{A} + \boldsymbol{B}) = 0 \tag{3.7}$$

The order of this real-valued polynomial is $2p$. The polynomial will as such have $2p$ roots λ_j, for $j = 1$ to $2p$, which are the eigenvalues of (3.6). For each of these eigenvalues a non-trivial solution $\boldsymbol{\psi}_j$ to (3.6) exists. This solution vector is called the eigenvector. The eigenvalues and therefore also the eigenvectors can either be real or complex. If all the eigenvalues are represented by complex conjugated pairs then the system is underdamped. This system behaviour will be assumed here, since it is the usual behaviour for a wide range of civil engineering structures. The complex conjugated pairs of eigenvalues are then given by

$$\begin{gathered} \lambda_j, \lambda_{j+1} = -\omega_j \zeta_j \pm i\omega_j \sqrt{1 - \zeta_j^2} = -2\pi f_j \zeta_j \pm i 2\pi f_j \sqrt{1 - \zeta_j^2} \\ \zeta_j < 1 \ , \quad j = 1,\ 3,\ \dots\ ,\ (2p-1) \end{gathered} \tag{3.8}$$

ω_j, f_j and ζ_j are the angular and natural eigenfrequencies, and the damping ratio, of the jth underdamped mode, respectively. From (3.2) and (3.3) it follows that the eigenvector $\boldsymbol{\psi}_j$ must have the form

$$\boldsymbol{\psi}_j = \begin{bmatrix} \boldsymbol{\Phi}_j \\ \lambda_j \boldsymbol{\Phi}_j \end{bmatrix} , \quad j = 1,\ 2,\ \dots\ ,\ 2p \tag{3.9}$$

and insertion of $\boldsymbol{A}$ and $\boldsymbol{B}$ and (3.9) into (3.6) yields

$$\left(\boldsymbol{M}\lambda_j^2 + \boldsymbol{C}\lambda_j + \boldsymbol{K} \right) \boldsymbol{\Phi}_j = \boldsymbol{0} \ , \quad j = 1,\ 2,\ \dots\ ,\ 2p \tag{3.10}$$

This is a standard eigenvalue problem for the second-order system, with $\boldsymbol{\Phi}_j$ being the non-trivial solution vector of it. This vector is also called the damped complex mode shape or simply the mode shape. Assembling all eigenvectors $\boldsymbol{\psi}_j$, defines the complex modal matrix

$$\boldsymbol{\Psi} = \begin{bmatrix} \boldsymbol{\Phi}_1 & \boldsymbol{\Phi}_2 & . \ . \ . & \boldsymbol{\Phi}_{2p} \\ \lambda_1 \boldsymbol{\Phi}_1 & \lambda_1 \boldsymbol{\Phi}_2 & . \ . \ . & \lambda_{2p} \boldsymbol{\Phi}_{2p} \end{bmatrix} \tag{3.11}$$

This matrix has the orthogonality properties

$$\boldsymbol{\Psi}^T \boldsymbol{A} \boldsymbol{\Psi} = \boldsymbol{M}_d \ , \quad \boldsymbol{\Psi}^T \boldsymbol{B} \boldsymbol{\Psi} = -\boldsymbol{\lambda} \boldsymbol{M}_d \ , \quad \left\{ \begin{aligned} \boldsymbol{\lambda} &= diag\{\lambda_j\} \\ \boldsymbol{M}_d &= diag\{m_j\} \end{aligned} \right. \tag{3.12}$$

with $\boldsymbol{\lambda}$ and $\boldsymbol{M}_d$ being diagonal matrices containing the $2p$ eigenvalues λ_j and the damped modal masses m_j, respectively.

For zero initial conditions, the solution of (3.1) is conveniently written in terms of the convolution integral as

$$\boldsymbol{z}(t) = \int_0^t \boldsymbol{h}(t-\tau)\boldsymbol{f}(\tau)\,d\tau\ , \qquad \begin{cases} \boldsymbol{z}(0) = \boldsymbol{0} \\ \dot{\boldsymbol{z}}(0) = \boldsymbol{0} \end{cases} \tag{3.13}$$

where $\boldsymbol{h}(t)$ is the $p \times p$ impulse response function. This function fully describes the dynamic behaviour of the structural system completely, and can conveniently be expressed in terms of the modal decomposed system as

$$\boldsymbol{h}(t) = \sum_{j=1}^{2p} \frac{\boldsymbol{\Phi}_j \boldsymbol{\Phi}_j^T}{m_j} e^{\lambda_j t} = \sum_{j=1}^{2p} \boldsymbol{R}_j e^{\lambda_j t} \tag{3.14}$$

m_j is the jth diagonal element of M_d, and R_j is the residue matrix that corresponds to the jth eigenvalue. Since $\boldsymbol{R}_j$ is a normalized matrix, it is insensitive to any scaling of the mode shapes.

3.1.3 A Canonical Realization of a Structural System

The system (3.3) is usually preferred in classical analytical vibration theory due to the assumption of the symmetric $\boldsymbol{M}$, $\boldsymbol{C}$ and $\boldsymbol{K}$ which also makes $\boldsymbol{A}$ and $\boldsymbol{B}$ symmetric. Symmetric $\boldsymbol{A}$ and $\boldsymbol{B}$ can simultaneously be reduced to diagonal forms. However, in the following sections the second order differential equation system is generalized to a system of arbitrary order. In this case the representation (3.3) is not suitable, and canonical realizations are preferred instead. Also, in the context of system identification the system needs to be modelled canonically in order to carry out the numerical manipulations. The system (3.3) can be converted to a canonical form by pre multiplication of the state equation of (3.3) with $\boldsymbol{A}^{-1}$ to yield

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{F}\boldsymbol{x}(t) + \boldsymbol{E}\boldsymbol{f}(t) \\ \boldsymbol{F} &= -\boldsymbol{A}^{-1}\boldsymbol{B} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{M}^{-1}\boldsymbol{K} & -\boldsymbol{M}^{-1}\boldsymbol{C} \end{bmatrix}, \quad \boldsymbol{E} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{M}^{-1} \end{bmatrix} \end{aligned} \tag{3.15}$$

Notice that $\boldsymbol{A}^{-1}$ and $\boldsymbol{B}$ can be written as

$$\boldsymbol{A}^{-1} = \boldsymbol{\Psi}\boldsymbol{M}_d^{-1}\boldsymbol{\Psi}^T\ , \qquad \boldsymbol{B} = -\boldsymbol{\Psi}^{-T}\boldsymbol{\lambda}\boldsymbol{M}_d\boldsymbol{\Psi}^{-1} \tag{3.16}$$

From these properties and (3.15), it is simple to verify that $\boldsymbol{\Psi}$ and its inverse diagonalize $\boldsymbol{F}$ as

$$\begin{aligned} \boldsymbol{F} &= -\boldsymbol{A}^{-1}\boldsymbol{B} \\ &= \boldsymbol{\Psi}\boldsymbol{M}_d^{-1}\boldsymbol{\Psi}^T\boldsymbol{\Psi}^{-T}\boldsymbol{\lambda}\boldsymbol{M}_d\boldsymbol{\Psi}^{-1} \\ &= \boldsymbol{\Psi}\boldsymbol{M}_d^{-1}\boldsymbol{\lambda}\boldsymbol{M}_d\boldsymbol{\Psi}^{-1} \\ &= \boldsymbol{\Psi}\boldsymbol{\lambda}\boldsymbol{\Psi}^{-1} \end{aligned} \tag{3.17}$$

with $\boldsymbol{\lambda}$ being the eigenvalues of $\boldsymbol{F}$. As seen the damped modal masses are eliminated from (3.17). This is because $\boldsymbol{K}$ and $\boldsymbol{C}$ in $\boldsymbol{F}$ have been normalised with $\boldsymbol{M}$, see (3.15). This implies that an arbitrary normalization of the eigenvectors can be applied without changing $\boldsymbol{F}$. Therefore, the damped modal mass is meaningless for this realization. The solution of (3.15) may still conveniently be described in terms of the convolution integral as in (3.13) by means of the impulse response function $\boldsymbol{h}(t)$, which is now defined as, see e.g. Kailath [48]

$$\boldsymbol{h}(t) = \sum_{j=1}^{2p} \boldsymbol{\Phi}_j \boldsymbol{\Psi}^{-1j} \boldsymbol{E} e^{\lambda_j t} = \sum_{j=1}^{2p} \boldsymbol{R}_j e^{\lambda_j t} \tag{3.18}$$

where $\boldsymbol{\Psi}^{-1j}$ signifies the jth row of $\boldsymbol{\Psi}^{-1}$. Therefore, the only difference in the definition of the impulse response function of the two realizations is the way the residue $\boldsymbol{R}_j$ is defined. It is no longer dependent on any normalization, which underlines that the modal mass is meaningless for this realization.

3.1.4 Spectrum Analysis

In this section, it is considered how to represent the dynamics of the structural system if the applied excitation $\boldsymbol{f}(t)$ is a stationary zero mean Gaussian distributed stochastic process. In this case the response $\boldsymbol{z}(t)$ of the system will also be a Gaussian distributed stochastic process, and it will also be stationary when the effects of the initial conditions have faded away. This section is primarily based on Bendat et al. [14], and it will be shown how stochastically excited systems can be studied in frequency domain. Since $\boldsymbol{f}(t)$ is assumed zero mean, it is fully described by its covariance function $\boldsymbol{\Pi}(\tau)$, defined as

$$\boldsymbol{\Pi}(\tau) = E\left[\boldsymbol{f}(t)\boldsymbol{f}^T(t-\tau)\right] \tag{3.19}$$

Due to the linearity of the system, the system response $\boldsymbol{z}(t)$ will also be fully described by its covariance function, which can be defined in terms of (3.13) and (3.19) as

$$\Gamma(\tau) = E\left[\boldsymbol{z}(t)\boldsymbol{z}^T(t-\tau)\right] = \int_0^\infty \int_0^\infty \boldsymbol{h}(t_1)\Pi(t_2 - t_1 + \tau)\boldsymbol{h}^T(t_1)\,dt_1\,dt_2 \qquad (3.20)$$

The covariance functions are time domain representations of the statistical properties of the stochastic processes. The processes $\boldsymbol{f}(t)$ and $\boldsymbol{z}(t)$ can equally well be represented in frequency domain by using the Wiener-Khinchine relation, see e.g. Bendat et al. [14], to yield

$$\boldsymbol{S}_{\boldsymbol{ff}}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Pi(\tau)e^{-i\omega\tau}d\tau \ , \quad \boldsymbol{S}_{\boldsymbol{zz}}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Gamma(\tau)e^{-i\omega\tau}d\tau \qquad (3.21)$$

$\boldsymbol{S}_{\boldsymbol{ff}}(\omega)$ and $\boldsymbol{S}_{\boldsymbol{zz}}(\omega)$ are the spectral density functions of $\boldsymbol{f}(t)$ and $\boldsymbol{z}(t)$ with ω being an arbitrary angular frequency. By introducing the frequency response function $\boldsymbol{H}(\omega)$ as the Fourier transformed impulse response function

$$\boldsymbol{H}(\omega) = \int_{-\infty}^{\infty} \boldsymbol{h}(\nu)e^{-i\omega\nu}d\nu \qquad (3.22)$$

it is possible to represent $\boldsymbol{S}_{\boldsymbol{zz}}(\omega)$ in terms of $\boldsymbol{S}_{\boldsymbol{ff}}(\omega)$ as

$$\boldsymbol{S}_{\boldsymbol{zz}}(\omega) = \boldsymbol{H}(\omega)\boldsymbol{S}_{\boldsymbol{ff}}(\omega)\boldsymbol{H}^H(\omega) \qquad (3.23)$$

where the superscript ^{H} signifies Hermitan or complex conjugate transpose. The frequency response function is directly related to the impulse response function, and can as such also be represented in terms of the residue $\boldsymbol{R}_j$ as

$$\boldsymbol{H}(\omega) = \sum_{j=1}^{2p} \frac{\boldsymbol{R}_j}{i\omega - \lambda_j} \qquad (3.24)$$

which follows directly by inserting (3.14) into (3.22). In some cases the spectral densities are easy to calculate. If $\boldsymbol{f}(t)$ is a Gaussian white noise, then its covariance function is especially simple

$$\Pi(\tau) = E\left[\boldsymbol{f}(t)\boldsymbol{f}^T(t-\tau)\right] = \boldsymbol{F}\delta(\tau) \qquad (3.25)$$

$\boldsymbol{F}$ is a constant intensity matrix and $\delta(\tau)$ is the Dirac delta function. The spectral density function of $\boldsymbol{f}(t)$ is constant and equal to

$$\boldsymbol{S}_{\boldsymbol{ff}} = \frac{1}{2\pi}\boldsymbol{F} \qquad (3.26)$$

The spectral density of the response of a Gaussian white noise excited second-order system is then obtained from (3.23) and (3.24) as

$$\boldsymbol{S}_{\boldsymbol{zz}}(\omega) = \frac{1}{2\pi}\boldsymbol{H}(\omega)\boldsymbol{F}\boldsymbol{H}^{H}(\omega) = \frac{1}{2\pi}\sum_{j=1}^{2p}\sum_{k=1}^{2p}\frac{\boldsymbol{R}_j\boldsymbol{F}\boldsymbol{R}_k^{H}}{(i\omega-\lambda_j)(i\omega-\lambda_k)} \tag{3.27}$$

The study of the spectral densities is also called spectrum analysis and is a helpful tool for modal analysis of stochastically excited systems.

3.2 Modelling of General Structural Systems

The use of the second-order differential equation system, for description of the dynamic behaviour of a structural system, only applies to systems where the number of observed mass points is equal to the total number of vibrating masses of the model. In the general case the number of observed mass points might be smaller than the total number of vibrating masses. In this case the mathematical model will not be the second order differential equation (3.1) but instead a differential equation of higher order. If for example the vibrations of a single mass point are observed in a system having p masses, then the order of the associated differential equation is $2p$.

3.2.1 Constructing a General Mathematical Model

From a system identification point of view a generalization of the mathematical model is therefore necessary, since the number of measurement channels is usually less than the number of identified modes. A generalized multivariate model can be formulated as in the following definition.

Definition 3.1 - Linear Time-Invariant Continuous-Time Structural System

Denote D the differential operator, defined as $D\boldsymbol{z}(t)=\dot{\boldsymbol{z}}(t)$. The p-variate continuous-time differential equation of motion of the structural system of order s is then given by

$$\begin{aligned} &D^{s}\boldsymbol{z}(t) + \boldsymbol{A}_{z,s-1}D^{s-1}\boldsymbol{z}(t) + \ldots + \boldsymbol{A}_{z,1}D\boldsymbol{z}(t) + \boldsymbol{A}_{z,0}\boldsymbol{z}(t) = \\ &\boldsymbol{B}_{f,s-2}D^{s-2}\boldsymbol{f}(t) + \ldots + \boldsymbol{B}_{f,1}D\boldsymbol{f}(t) + \boldsymbol{B}_{f,0}\boldsymbol{f}(t) \end{aligned} \tag{3.28}$$

where the matrices $\boldsymbol{A}_{z,i}$ and $\boldsymbol{B}_{f,i}$ are all of the dimension $p \times p$. The displacement vector $\boldsymbol{z}(t)$ and its derivatives are all of the dimension $p \times 1$. The $p \times 1$ vector $\boldsymbol{f}(t)$ describes the forces applied to the system. □

The modes of a structural system will typically be underdamped, which implies that each mode is described by a pair of complex conjugated eigenvalues. In this situation the order s will be defined as $s = \frac{2N}{p}$ with N being the number of underdamped modes.

The number of output channels must be selected so that s becomes an integer value. The matrices $\boldsymbol{A}_{z,i}$ are now the system matrices that describe the vibration of the mass points of the model. The matrices $\boldsymbol{B}_{f,i}$ describes the coupling to the unobserved vibrating mass points of the model. For the second order structural differential system (3.1) it can be verified that only $\boldsymbol{B}_{f,0}$ exist and that it is equal to $\boldsymbol{M}^{-1}$. However, since it is assumed that the system is subjected to ambient excitation, it still remains to model this unknown excitation.

3.2.2 Modelling of Ambient Excitation

Instead of obtaining the ambient excitation by indirect measurements such as the sea state, it is often assumed that the ambient excitation is given as the output of a linear time-invariant shaping filter subjected to Gaussian white noise. Due to the Gaussian assumption, it is implicitly assumed that the true ambient excitation is at least weakly stationary. If the ambient excitation can be described by filtered white noise, then it is possible to derive a model for it. The above assumptions lead to the following definition on how to model the stochastic excitation $\boldsymbol{f}(t)$ applied to the structural system (3.28).

Definition 3.2 - Excitation Generated by a Shaping Filter Subjected to White Noise

Assume that the excitation $\boldsymbol{f}(t)$ of the structural system is obtained as the output of an mth-order p-variate linear time-invariant continuous-time shaping filter

$$D^m\boldsymbol{f}(t) + \boldsymbol{A}_{f,m-1}D^{m-1}\boldsymbol{f}(t) + \ \ldots \ + \boldsymbol{A}_{f,1}D\boldsymbol{f}(t) + \boldsymbol{A}_{f,0}\boldsymbol{f}(t) = \boldsymbol{w}(t) \tag{3.29}$$

For simplicity, it is assumed that $\boldsymbol{f}(t)$ and $\boldsymbol{w}(t)$ have the same dimensions as $\boldsymbol{z}(t)$. This implies that the matrices $\boldsymbol{A}_{f,i}$ all have the dimension $p \times p$. The stochastic process $\boldsymbol{w}(t)$ is a zero-mean Gaussian white noise, fully described by its covariance function. This covariance function is defined in terms of the $p \times p$ intensity matrix $\boldsymbol{W}$ as

$$E\left[\boldsymbol{w}(t)\right] = \boldsymbol{0} \ , \quad E\left[\boldsymbol{w}(t)\boldsymbol{w}^T(t-\tau)\right] = \delta(\tau)\boldsymbol{W} \tag{3.30}$$

where $\delta(\tau)$ is the Dirac delta function. These statistical properties will in the following be abbreviated $NID(\boldsymbol{0},\boldsymbol{W})$. □

It is obvious that the response $\boldsymbol{z}(t)$ of the system will contain a mixture of the dynamic behaviour of the structural system and of the excitation. It is also intuitively clear that during a system identification the dynamic modes of this filter will also be estimated. These modes are, together with any noise modes, called nonphysical modes. In this way they can be distinguished from the physical modes of the structural system. In figure 3.1, the generation of the excitation from Gaussian white noise is explained in terms of spectral densities of a univariate system.

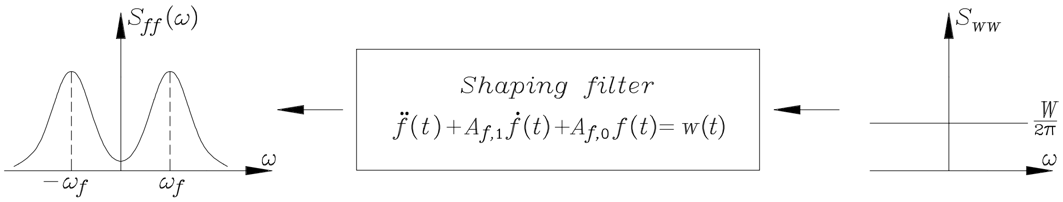

Figure 3.1: The constant spectral density of the univariate Gaussian white noise $w(t)$ with an intensity W is shaped into the desired univariate spectral density of $f(t)$. This is done by means of the SDOF shaping filter having ω_f as angular eigenfrequency.

If the structural system can be combined with the shaping filter of the excitation into a resulting linear system subjected to a Gaussian white noise, then it is possible to represent this system by a discrete-time Gaussian white noise excited state space realization or an equivalent ARMAV model, according to section 2.3. The next natural step is therefore to combine the structural system and the shaping filter of the excitation.

3.3 Combined Continuous-time Systems

The response $\boldsymbol{z}(t)$ from the structural system can be expressed directly in terms of the Gaussian white noise $\boldsymbol{w}(t)$ by a multivariate convolution of (3.28) into (3.29). This convolution procedure can be represented in a simple manner using a state space approach. The generalized structural system and the shaping filter can then conveniently be represented by the following two coupled state space realizations

$$\begin{aligned} \dot{\boldsymbol{x}}_1(t) &= \boldsymbol{A}_1\boldsymbol{x}_1(t) + \boldsymbol{B}_1\boldsymbol{f}(t) \\ \boldsymbol{z}(t) &= \boldsymbol{H}_1\boldsymbol{x}_1(t) \end{aligned} \tag{3.31}$$

$$\begin{aligned} \dot{\boldsymbol{x}}_2(t) &= \boldsymbol{A}_2\boldsymbol{x}_2(t) + \boldsymbol{B}_2\boldsymbol{w}(t) \ , \quad \boldsymbol{w}(t) \in NID(\boldsymbol{0},\boldsymbol{W}) \\ \boldsymbol{f}(t) &= \boldsymbol{H}_2\boldsymbol{x}_2(t) \end{aligned} \tag{3.32}$$

with the matrix triples $\{\boldsymbol{A}_1, \boldsymbol{B}_1, \boldsymbol{H}_1\}$ and $\{\boldsymbol{A}_2, \boldsymbol{B}_2, \boldsymbol{H}_2\}$ given by

$$\boldsymbol{A}_1 = \begin{bmatrix} \mathbf{0} & \boldsymbol{I} & . & . & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & . & . & \mathbf{0} & \mathbf{0} \\ . & . & . & . & . & \\ . & . & . & . & . & \\ \mathbf{0} & \mathbf{0} & . & . & \boldsymbol{I} & \mathbf{0} \\ -\boldsymbol{A}_{z,0} & -\boldsymbol{A}_{z,1} & . & . & -\boldsymbol{A}_{z,s-2} & -\boldsymbol{A}_{z,s-1} \end{bmatrix}$$

$$\boldsymbol{B}_1 = \begin{bmatrix} \boldsymbol{I} & \mathbf{0} & . & . & \mathbf{0} & \mathbf{0} \\ \boldsymbol{A}_{z,s-1} & \boldsymbol{I} & . & . & \mathbf{0} & \mathbf{0} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \boldsymbol{A}_{z,2} & \boldsymbol{A}_{z,3} & . & . & \boldsymbol{I} & \mathbf{0} \\ \boldsymbol{A}_{z,1} & \boldsymbol{A}_{z,2} & . & . & \boldsymbol{A}_{z,s-1} & \boldsymbol{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ \boldsymbol{B}_{f,s-2} \\ . \\ . \\ \boldsymbol{B}_{f,1} \\ \boldsymbol{B}_{f,0} \end{bmatrix} \tag{3.33}$$

$$\boldsymbol{H}_1 = \begin{bmatrix} \boldsymbol{I} & \mathbf{0} & . & . & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$\boldsymbol{A}_2 = \begin{bmatrix} \mathbf{0} & \boldsymbol{I} & . & . & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & . & . & \mathbf{0} & \mathbf{0} \\ . & . & . & . & . & \\ . & . & . & . & . & \\ \mathbf{0} & \mathbf{0} & . & . & \boldsymbol{I} & \mathbf{0} \\ -\boldsymbol{A}_{f,0} & -\boldsymbol{A}_{f,1} & . & . & -\boldsymbol{A}_{f,m-2} & -\boldsymbol{A}_{f,m-1} \end{bmatrix}$$

$$\boldsymbol{B}_2 = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ . \\ . \\ \mathbf{0} \\ \boldsymbol{I} \end{bmatrix} \tag{3.34}$$

$$\boldsymbol{H}_2 = \begin{bmatrix} \boldsymbol{I} & \mathbf{0} & . & . & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

These realizations are in observability canonical form and their construction follows the principles shown for the discrete-time systems in section 2.5, see also Kailath [48]

Introduce the following augmented state vector

$$\boldsymbol{x}_c(t) = \begin{bmatrix} \boldsymbol{x}_1(t) \\ \boldsymbol{x}_2(t) \end{bmatrix} \tag{3.35}$$

If $\boldsymbol{f}(t)$ in (3.31) is substituted by the observation equation of (3.32), and the resulting two state equations of (3.31) and (3.32) are stacked, then the following coupled set of equations is obtained

$$\begin{aligned} \dot{\boldsymbol{x}}_1(t) &= \boldsymbol{A}_1\boldsymbol{x}_1(t) + \boldsymbol{B}_1\boldsymbol{H}_2\boldsymbol{x}_2(t) \\ \dot{\boldsymbol{x}}_2(t) &= \boldsymbol{A}_2\boldsymbol{x}_2(t) + \boldsymbol{B}_2\boldsymbol{w}(t) \end{aligned} \tag{3.36}$$

Combine this system with the augmented state vector and the observation equation of (3.31) to yield

$$\begin{aligned} &\dot{\boldsymbol{x}}_c(t) = \boldsymbol{F}_c\boldsymbol{x}_c(t) + \boldsymbol{B}_c\boldsymbol{w}(t) \;, \quad \boldsymbol{w}(t) \in NID(\boldsymbol{0},\boldsymbol{W}) \\ &\boldsymbol{z}(t) = \boldsymbol{H}_c\boldsymbol{x}_c(t) \\ &\boldsymbol{F}_c = \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{B}_1\boldsymbol{H}_2 \\ \boldsymbol{0} & \boldsymbol{A}_2 \end{bmatrix}, \quad \boldsymbol{B}_c = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{B}_2 \end{bmatrix}, \quad \boldsymbol{H}_c = \begin{bmatrix} \boldsymbol{H}_1 & \boldsymbol{0} \end{bmatrix} \end{aligned} \tag{3.37}$$

As seen, this convolution follows the procedures described in section 2.1.2 for nonwhite excited linear and time-invariant discrete-time systems. The dimensions of $\boldsymbol{F}_c$, $\boldsymbol{B}_c$ and $\boldsymbol{H}_c$ are:

$$\begin{aligned} dim\,\boldsymbol{F}_c &= (s+m)p \times (s+m)p \\ dim\,\boldsymbol{B}_c &= (s+m)p \times p \\ dim\,\boldsymbol{H}_c &= p \times (s+m)p \end{aligned} \tag{3.38}$$

which implies that the order of the homogenous part of the corresponding multivariate differential equation is $n = s+m$. This differential equation will have the form

$$\begin{aligned} &D^n\boldsymbol{z}(t) + \boldsymbol{C}_{z,n-1}D^{n-1}\boldsymbol{z}(t) + \ldots + \boldsymbol{C}_{z,1}D\boldsymbol{z}(t) + \boldsymbol{C}_{z,0}\boldsymbol{z}(t) = \\ &\boldsymbol{C}_{w,s-2}D^{s-2}\boldsymbol{w}(t) + \ldots + \boldsymbol{C}_{w,1}D\boldsymbol{w}(t) + \boldsymbol{C}_{w,0}\boldsymbol{w}(t) \\ &\boldsymbol{w}(t) \in NID(\boldsymbol{0},\boldsymbol{W}) \end{aligned} \tag{3.39}$$

Based on theorem 2.2, which also applies for continuous-time state space systems, see Gohberg et al. [30], the matrix coefficients $\boldsymbol{C}_{z,i}$ and $\boldsymbol{C}_{w,i}$ can be obtained by

$$\begin{aligned} \begin{bmatrix} \boldsymbol{C}_{z,0} & \boldsymbol{C}_{z,1} & . & . & \boldsymbol{C}_{z,n-1} \end{bmatrix} &= -\boldsymbol{H}_c \boldsymbol{F}_c^n \boldsymbol{Q}_o^{-1}(n) \\ \begin{bmatrix} \boldsymbol{C}_{w,0} & . & . & \boldsymbol{C}_{w,s-2} & \boldsymbol{0} & . & . & \boldsymbol{0} \end{bmatrix} &= \begin{bmatrix} \boldsymbol{C}_{z,0} & \boldsymbol{C}_{z,1} & . & . & \boldsymbol{C}_{z,n-1} & \boldsymbol{I} \end{bmatrix} \boldsymbol{T}(n+1) \end{aligned} \tag{3.40}$$

with $\boldsymbol{Q}_o(n)$ being the non-singular observability matrix, based on $\{\boldsymbol{F}_c, \boldsymbol{H}_c\}$. The definition of the matrix $\boldsymbol{T}(n+1)$ follows theorem 2.2 as

$$\boldsymbol{T}(n+1) = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & . & . & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{H}_c\boldsymbol{B}_c & \boldsymbol{0} & . & . & \boldsymbol{0} & \boldsymbol{0} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \boldsymbol{H}_c\boldsymbol{F}_c^{n-2}\boldsymbol{B}_c & \boldsymbol{H}_c\boldsymbol{F}_c^{n-3}\boldsymbol{H}_c & . & . & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{H}_c\boldsymbol{F}_c^{n-1}\boldsymbol{B}_c & \boldsymbol{H}_c\boldsymbol{F}_c^{n-2}\boldsymbol{B}_c & . & . & \boldsymbol{H}_c\boldsymbol{B}_c & \boldsymbol{0} \end{bmatrix} \tag{3.41}$$

If the following linear transformations are applied

$$\boldsymbol{x}(t) = \boldsymbol{T}\boldsymbol{x}_c(t), \quad \boldsymbol{F} = \boldsymbol{T}\boldsymbol{F}_c\boldsymbol{T}^{-1}, \quad \boldsymbol{B} = \boldsymbol{T}\boldsymbol{B}_c, \quad \boldsymbol{H} = \boldsymbol{H}_c\boldsymbol{T}^{-1} \tag{3.42}$$

where $\boldsymbol{T} = \boldsymbol{Q}_o(n)$, then (3.37), or equivalently (3.39), can be represented by the following observability canonical state space realization

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{F}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{w}(t), \quad \boldsymbol{w}(t) \in NID(\boldsymbol{0}, \boldsymbol{W}) \\ \boldsymbol{z}(t) &= \boldsymbol{H}\boldsymbol{x}(t) \end{aligned} \tag{3.43}$$

The state vector $\boldsymbol{x}(t)$ of this realization will then be defined as

$$\boldsymbol{x}(t) = \begin{bmatrix} \boldsymbol{z}(t) \\ D\boldsymbol{z}(t) \\ . \\ . \\ D^{n-2}\boldsymbol{z}(t) \\ D^{n-1}\boldsymbol{z}(t) \end{bmatrix} \tag{3.44}$$

and the matrix triple $\{\boldsymbol{F}, \boldsymbol{B}, \boldsymbol{H}\}$ as

$$
\boldsymbol{F} = \begin{bmatrix} \mathbf{0} & \boldsymbol{I} & . & . & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & . & . & \mathbf{0} & \mathbf{0} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \mathbf{0} & \mathbf{0} & . & . & \mathbf{0} & \boldsymbol{I} \\ -\boldsymbol{C}_{z,0} & -\boldsymbol{C}_{z,1} & . & . & -\boldsymbol{C}_{z,n-2} & -\boldsymbol{C}_{z,n-1} \end{bmatrix}
$$

$$
\boldsymbol{B} = \begin{bmatrix} \boldsymbol{I} & \mathbf{0} & . & . & \mathbf{0} & \mathbf{0} \\ \boldsymbol{C}_{z,n-1} & \boldsymbol{I} & . & . & \mathbf{0} & \mathbf{0} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \boldsymbol{C}_{z,2} & \boldsymbol{C}_{z,3} & . & . & \boldsymbol{I} & \mathbf{0} \\ \boldsymbol{C}_{z,1} & \boldsymbol{C}_{z,2} & . & . & \boldsymbol{C}_{z,n-1} & \boldsymbol{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ . \\ . \\ \mathbf{0} \\ \boldsymbol{C}_{w,s-2} \\ . \\ . \\ \boldsymbol{C}_{w,0} \end{bmatrix} \tag{3.45}
$$

$$
\boldsymbol{H} = \begin{bmatrix} \boldsymbol{I} & \mathbf{0} & . & . & \mathbf{0} & \mathbf{0} \end{bmatrix}
$$

The combination of the shaping filter and the structural system is in figure 3.2 explained in terms of spectral densities for an univariate system.

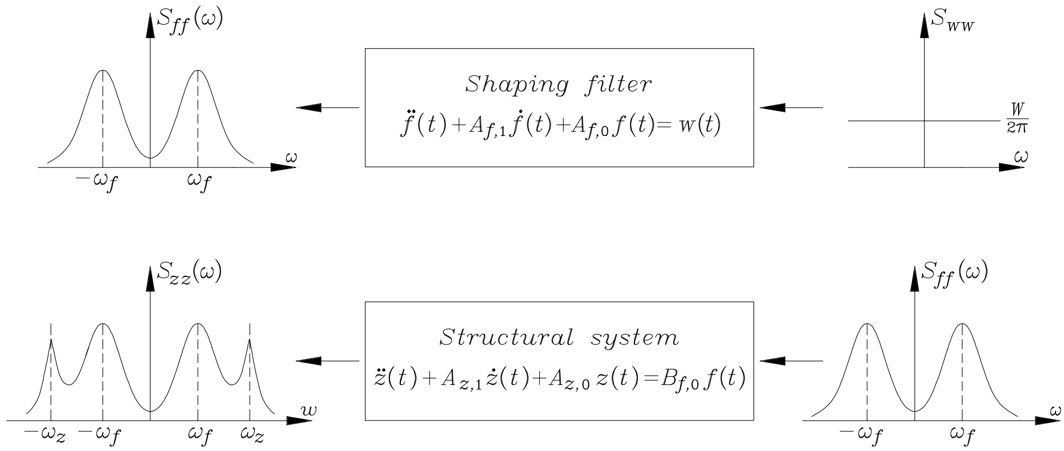

Figure 3.2: The constant spectral density of the univariate Gaussian white noise $w(t)$ with an intensity W is shaped into the desired univariate spectral density of $f(t)$ by means of the SDOF shaping filter. The angular eigenfrequency of the shaping filter is ω_f. The spectral density of $f(t)$ is then reshaped by means of the SDOF structural system to yield the spectral density of the output $z(t)$. The angular eigenfrequency of the structural system is ω_z.

A frequency domain interpretation can be found in Asmussen et al. [10] and Ibrahim et al. [41], where the multivariate shaping filter is referred to as a pseudo force system.

3.3.1 Generalization to an Arbitrary System Output

The realization (3.43) is an efficient way of representing the combined differential equation system (3.39). However, in some cases it is desirable to observe other characteristics than the displacements $\boldsymbol{z}(t)$ of the system. This can be accomplished by a simple modification of the observation equation of (3.43). The matrix $\boldsymbol{C}$ is defined as a generalized observation matrix and $\boldsymbol{y}(t)$ is a generalized output vector. Assume that $\boldsymbol{y}(t)$ is obtained as a linear combination of the states of $\boldsymbol{x}(t)$, and that this linear combination is controlled by the observation matrix $\boldsymbol{C}$. A state space realization of the generalized combined continuous-time system is then given by

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{F}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{w}(t) \; , \qquad \boldsymbol{w}(t) \in NID(\boldsymbol{0}, \boldsymbol{W}) \\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t) \end{aligned} \tag{3.46}$$

It should be noted that the state equation is unaffected, and that (3.43) can be recovered from setting $\boldsymbol{C} = \boldsymbol{H}$ and $\boldsymbol{y}(t) = \boldsymbol{z}(t)$. Three special cases are worth mentioning. These show the appearance of $\boldsymbol{C}$ in case the vector $\boldsymbol{y}(t)$ describes the displacements, the velocities, or the accelerations of the structural system:

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & . & . & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \boldsymbol{x}(t) = \boldsymbol{z}(t) \tag{3.47}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} & . & . & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \boldsymbol{x}(t) = \dot{\boldsymbol{z}}(t) \tag{3.48}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} & . & . & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \boldsymbol{x}(t) = \ddot{\boldsymbol{z}}(t) \tag{3.49}$$

Relation (3.49) implies that in order to extract the accelerations from the system, the order of (3.39) must be at least three. A way to reduce the necessary order is to include derivatives of the state vector $\boldsymbol{x}(t)$ in the observation equation in (3.45). However, in the present context this is not desirable, since this will introduce a Gaussian white noise component as a direct term in the observation equation. Such a component will make the system impossible to sample, since the zero-lag covariance of the output $\boldsymbol{y}(t)$ then will approach infinity, see e.g. Shats et al. [102]. In the following the construction of a univariate second-order structural system will be illustrated in two examples. In the first example the applied excitation is assumed to be Gaussian white noise. In this case the combined system resembles the second-order structural system described in section 3.1. The next example illustrates what happens when the excitation is no longer a Gaussian white noise.

3.3.2 Example 3.1: A White Noise Excited System

Consider a univariate structural system. The motion of this system is assumed to be described by the following second-order differential equation

$$\ddot{z}(t) + \frac{c}{m}\dot{z}(t) + \frac{k}{m}z(t) = \frac{1}{m}f(t) \tag{3.50}$$

m, *c* and *k* are the mass, viscous damping, and stiffness respectively. $\boldsymbol{z}(t)$ is the displacement of the system due the Gaussian distributed excitation $f(t) = w(t)$. This excitation is assumed to be a zero-mean Gaussian white noise described by the intensity *W*. Assume that the observed response is the displacement. Then the matrix triple $\{\boldsymbol{F}, \boldsymbol{B}, \boldsymbol{C}\}$of the combined continuous-time system (3.46) is given by

$$\boldsymbol{F} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{3.51}$$

The observability matrix $\boldsymbol{Q}_o(2)$ obtained from $\{\boldsymbol{F}, \boldsymbol{C}\}$is simply an 2×2 identity matrix. □

3.3.3 Example 3.2: A Nonwhite Excited System

Consider the univariate structural system in example 3.1. The motion of this system was described by the second-order differential equation (3.50). Assume now that the excitation $f(t)$ is a nonwhite Gaussian process, obtained by filtering a zero-mean Gaussian white noise $w(t)$, described the intensity *W*, through the second-order shaping filter

$$\ddot{f}(t) + a_1\dot{f}(t) + a_0 f(t) = w(t) \;, \quad w(t) \in NID(0, W) \tag{3.52}$$

From (3.33) and (3.34) the matrix triples $\{\boldsymbol{A}_1, \boldsymbol{B}_1, \boldsymbol{H}_1\}$ and $\{\boldsymbol{A}_2, \boldsymbol{B}_2, \boldsymbol{H}_2\}$ appear as

$$\begin{aligned} \boldsymbol{A}_1 &= \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}, \quad \boldsymbol{B}_1 = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}, \quad \boldsymbol{C}_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \\ \boldsymbol{A}_2 &= \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}, \quad \boldsymbol{B}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{C}_2 = \begin{bmatrix} 1 & 0 \end{bmatrix} \end{aligned} \tag{3.53}$$

and from (3.53), the matrix triple $\{\boldsymbol{F}_c, \boldsymbol{B}_c, \boldsymbol{H}_c\}$ of the combined continuous-time system (3.37), is given by

$$\boldsymbol{F}_c = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{m} & -\frac{c}{m} & \frac{1}{m} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -a_0 & -a_1 \end{bmatrix}, \quad \boldsymbol{B}_c = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{H}_c = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \tag{3.54}$$

The observability matrix $\boldsymbol{Q}_o(4)$ is obtained from $\{\boldsymbol{F}_c, \boldsymbol{H}_c\}$ to yield

$$\boldsymbol{Q}_o(4) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{k}{m} & -\frac{c}{m} & \frac{1}{m} & 0 \\ \frac{ck}{m^2} & -\frac{k}{m}+\frac{c^2}{m^2} & -\frac{c}{m} & \frac{1}{m} \end{bmatrix} \tag{3.55}$$

Assuming that the output $\boldsymbol{y}(t)$ is the displacement of the system, the combined system can be represented by the state space realization (3.46), defined by the matrices

$$\begin{aligned} \boldsymbol{F} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{a_0 k}{m} & -\frac{a_1 k + a_0 c}{m} & -\frac{k + a_1 c + a_0 m}{m} & -\frac{c + a_1 m}{m} \end{bmatrix} \\ \boldsymbol{B} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{m} \end{bmatrix} \\ \boldsymbol{C} &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \end{aligned} \tag{3.56}$$

□

3.4 Modal Decomposition of Combined Continuous-Time Systems

In this section the modal decomposition of the combined structural system is considered. The modal decomposition of this system clearly shows the problems exhibited when structural systems are identified from response measurements only. In such systems there will always be a mixture of modes originating from the structural system and the excitation. The question is whether this mixture affects the structural modes in some ways or not. It might also be that the measured data are either velocity or accelerations instead of displacements. Again the question is how this affects the identification of the structural modes. This section starts by modal decomposition of the combined continuous-time structural system. It is obvious that the eigenvalues of the combined continuous-time system must be the eigenvalues of the structural system and of the shaping filter. However, the question is whether or not the mode shapes are affected by the choice of output and by the convolution of the structural system and the shaping filter. Therefore, the emphasis is placed on analysing how the mode shapes of the structural system are affected by way the combined system is obtained.

3.4.1 Modal Decomposition of the Combined System

The free vibrations of the structural system are, in principle, obtained from the homogeneous part of the p-variate differential equation (3.45) of the structural system

$$D^s\mathbf{z}(t) + \boldsymbol{A}_{s-1}D^{s-1}\mathbf{z}(t) + \ldots + \boldsymbol{A}_1D\mathbf{z}(t) + \boldsymbol{A}_0\mathbf{z}(t) = \mathbf{0} \tag{3.57}$$

However, there is a problem since the dynamics of the structural system is mixed up with the dynamics of the excitation. The modal decomposition can as such not be applied to (3.57), but must be applied to the homogeneous part of the combined continuous-time system (3.39) instead, i.e. to

$$D^n\mathbf{z}(t) + \boldsymbol{C}_{n-1}D^{n-1}\mathbf{z}(t) + \ldots + \boldsymbol{C}_1D\mathbf{z}(t) + \boldsymbol{C}_0\mathbf{z}(t) = \mathbf{0} \tag{3.58}$$

Since $s \le n$ there will be situations where only a subset of the modes of the combined system originates from the structural dynamics. The modal decomposition of the combined continuous-time system is obtained from the homogeneous part of the state equation in (3.46)

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{F}\boldsymbol{x}(t) \tag{3.59}$$

with $\boldsymbol{x}(t)$ and $\boldsymbol{F}$ defined by (3.44) and (3.45). As in section 3.1.2, the solution form is assumed as

$$\boldsymbol{x}(t) = \boldsymbol{\psi}e^{\lambda t} \tag{3.60}$$

where $\boldsymbol{\psi}$ is a complex vector of the dimension $np \times 1$ and λ is a complex constant. Insertion of (3.60) into (3.59) shows that (3.60) is only a solution if and only if $\boldsymbol{\psi}$ is a solution to the first-order eigenvalue problem

$$(\boldsymbol{I}\lambda - \boldsymbol{F})\boldsymbol{\psi} = \mathbf{0} \tag{3.61}$$

This eigenvalue problem has only non-trivial solutions if the characteristic polynomial, given by

$$det(\boldsymbol{I}\lambda - \boldsymbol{F}) = 0 \tag{3.62}$$

is satisfied. The order of this real-valued characteristic polynomial is np. Thus, there will be np roots λ_j, which are the eigenvalues of $\boldsymbol{F}$. For each of these eigenvalues a non-trivial solution vector $\boldsymbol{\psi}_j$ exists, which are the corresponding eigenvector. In view of the structure of $\boldsymbol{x}(t)$ given by (3.44), it follows that $\boldsymbol{\psi}_j$ must have the form

$$\psi_j = \begin{bmatrix} \mathbf{\Phi}_j \\ \lambda_j \mathbf{\Phi}_j \\ . \\ . \\ \lambda_j^{n-2}\mathbf{\Phi}_j \\ \lambda_j^{n-1}\mathbf{\Phi}_j \end{bmatrix} , \quad j = 1,\ 2,\ \dots\ ,\ np \tag{3.63}$$

Insertion of $\boldsymbol{F}$ and (3.63) into (3.46) yields

$$\left(\boldsymbol{I}\lambda_j^n + \boldsymbol{C}_{n-1}\lambda_j^{n-1} + \dots + \boldsymbol{C}_1\lambda_j + \boldsymbol{C}_0\right)\mathbf{\Phi}_j = \mathbf{0} \tag{3.64}$$

which is an nth order eigenvalue problem, with $\mathbf{\Phi}_j$ being a non-trivial solution vector of it. Thus, for the jth of the sp eigenvalues of the underdamped structural system $\mathbf{\Phi}_j$ is actually the associated complex mode shape. By use of the modal matrix $\mathbf{\Psi}$, and its inverse, $\boldsymbol{F}$ can be diagonalized as

$$\mathbf{\Psi}^{-1}\boldsymbol{F}\mathbf{\Psi} = \boldsymbol{\lambda} \ , \quad \boldsymbol{\lambda} = diag\{\lambda_j\} \tag{3.65}$$

where $\boldsymbol{\lambda}$ is a diagonal matrix of np eigenvalues λ_j. The complex modal matrix is formed by all np eigenvectors as

$$\mathbf{\Psi} = \begin{bmatrix} \mathbf{\Phi}_1 & \mathbf{\Phi}_2 & . & . & \mathbf{\Phi}_{np-1} & \mathbf{\Phi}_{np} \\ \lambda_1\mathbf{\Phi}_1 & \lambda_2\mathbf{\Phi}_2 & . & . & \lambda_{np-1}\mathbf{\Phi}_{np-1} & \lambda_{np}\mathbf{\Phi}_{np} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \lambda_1^{n-2}\mathbf{\Phi}_1 & \lambda_2^{n-2}\mathbf{\Phi}_2 & . & . & \lambda_{np-1}^{n-2}\mathbf{\Phi}_{np-1} & \lambda_{np}^{n-2}\mathbf{\Phi}_{np} \\ \lambda_1^{n-1}\mathbf{\Phi}_1 & \lambda_2^{n-1}\mathbf{\Phi}_2 & . & . & \lambda_{np-1}^{n-1}\mathbf{\Phi}_{np-1} & \lambda_{np}^{n-1}\mathbf{\Phi}_{np} \end{bmatrix} \tag{3.66}$$

The modal decomposition of $\boldsymbol{F}$ is then trivially obtained as

$$\boldsymbol{F} = \mathbf{\Psi}\boldsymbol{\lambda}\mathbf{\Psi}^{-1} \tag{3.67}$$

It should be noticed that the np eigenvectors, and as such also the sp mode shapes of the structural modes, may be arbitrarily scaled without affecting the modal decomposition. For this reason the mode shapes obtained from this modal decomposition are called scaled mode shapes. It has been assumed that all *np* eigenvalues are distinct, since this assumption is needed for $\boldsymbol{F}$ to be diagonalizable. When the eigenvalues are not distinct, $\boldsymbol{\lambda}$ will not be a diagonal matrix, but have diagonal blocks for distinct eigenvalues and Jordan blocks for repeated ones. $\boldsymbol{\Psi}$ will in this case consist of eigenvectors and generalized eigenvectors, see e.g. Gohberg et al. [30] and Metha et al. [78].

3.4.2 Effects of Arbitrarily Chosen Output

The free vibrations of the state space system (3.46) are given by

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{F}\boldsymbol{x}(t) \\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t) \end{aligned} \tag{3.68}$$

The scaled mode shapes of the previous section were obtained under the assumption that the output of this system was the displacement $\boldsymbol{z}(t)$ of the mass points. Thus, by using the observation equation in (3.68) with $\boldsymbol{C} = \boldsymbol{H}$, the extraction of the scaled mode shape $\boldsymbol{\Phi}_j$ may be written as

$$\boldsymbol{\Phi}_j = \boldsymbol{H}\boldsymbol{\psi}_j \ , \quad j = 1,\ 2,\ \ldots\ ,\ sp \tag{3.69}$$

However, from a system identification point of view, it is interesting to investigate how the mode shapes are affected by the choice of output, since the measured response of a civil engineering structure is often its accelerations. For a general choice of observation matrix $\boldsymbol{C}$, define temporarily the generalized mode shapes $\boldsymbol{\varphi}_j$. These mode shapes are then obtained as

$$\boldsymbol{\varphi}_j = \boldsymbol{C}\boldsymbol{\psi}_j \ , \quad j = 1,\ 2,\ \ldots\ ,\ sp \tag{3.70}$$

This relation clearly shows that the generalized mode shapes certainly depends on the choice of output $\boldsymbol{y}(t)$, and that they in some cases will be scaled versions of $\boldsymbol{\Phi}_j$. Consider the three special cases (3.47), (3.48) and (3.49). It follows from (3.70) that if $\boldsymbol{y}(t)$ is the displacement of the system then

$$\boldsymbol{\varphi}_j = \boldsymbol{C}\boldsymbol{\psi}_j = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & . & . & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}\boldsymbol{\psi}_j = \boldsymbol{\Phi}_j \ , \quad j = 1,\ 2,\ \ldots\ ,\ sp \tag{3.71}$$

This result is of course equivalent to (3.69). If $\boldsymbol{y}(t)$ is the velocity of the system then

$$\boldsymbol{\varphi}_j = \boldsymbol{C}\boldsymbol{\psi}_j = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} & . & . & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}\boldsymbol{\psi}_j = \boldsymbol{\Phi}_j\lambda_j \ , \quad j = 1,\ 2,\ \ldots\ ,\ sp \tag{3.72}$$

and finally, if $\boldsymbol{y}(t)$ is the acceleration then

$$\boldsymbol{\varphi}_j = \boldsymbol{C}\boldsymbol{\psi}_j = \begin{bmatrix}\mathbf{0} & \mathbf{0} & \boldsymbol{I} & . & . & \mathbf{0} & \mathbf{0}\end{bmatrix}\boldsymbol{\psi}_j = \boldsymbol{\Phi}_j\lambda_j^2 \ , \quad j = 1,\ 2,\ \ldots\ ,\ sp \tag{3.73}$$

It is seen that all mode shapes $\boldsymbol{\Phi}_j$ are individually scaled due to change of $\boldsymbol{C}$, but since the mode shapes $\boldsymbol{\Phi}_j$ are already arbitrarily scaled this additional scaling is immaterial for the modal decomposition. Therefore, there is no point in distinguishing between $\boldsymbol{\varphi}_j$ and $\boldsymbol{\Phi}_j$, and (3.70) can just as well be written in terms of $\boldsymbol{\Phi}_j$ as

$$\boldsymbol{\Phi}_j = \boldsymbol{C}\boldsymbol{\psi}_j \ , \quad j = 1,\ 2,\ \ldots\ ,\ sp \tag{3.74}$$

Finally, the question is whether or not the convolution of the structural system and the shaping filter of the excitation have affected the scaled mode shapes of the structural system or not.

3.4.3 Effects of the Shaping Filter Convolution

Assume that the observed output is the displacement $\boldsymbol{z}(t)$ of the system, and consider the free vibrations of the combined system (3.37)

$$\begin{aligned} \dot{\boldsymbol{x}}_c(t) &= \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{B}_1\boldsymbol{H}_2 \\ \mathbf{0} & \boldsymbol{A}_2 \end{bmatrix} \boldsymbol{x}_c(t) \\ \boldsymbol{z}(t) &= \begin{bmatrix} \boldsymbol{H}_1 & \mathbf{0} \end{bmatrix} \boldsymbol{x}_c(t) \end{aligned} \tag{3.75}$$

The structural system is described by the matrix triple $\{\boldsymbol{A}_1, \boldsymbol{B}_1, \boldsymbol{H}_1\}$ see (3.33), whereas $\{\boldsymbol{A}_2, \boldsymbol{H}_2\}$ relates to the shaping filter of the excitation, see (3.34). Since this realization is similar to (3.46) the np eigenvalues λ_j obtained from (3.62) also satisfies the characteristic polynomial

$$det\left(\lambda \boldsymbol{I} - \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{B}_1\boldsymbol{H}_2 \\ \mathbf{0} & \boldsymbol{A}_2 \end{bmatrix} \right) = 0 \tag{3.76}$$

and it is thus possible to find np non-trivial eigenvectors $\boldsymbol{\psi}_{j,c}$ that satisfy the eigenvalue problem

$$\left(\lambda_j \boldsymbol{I} - \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{B}_1\boldsymbol{H}_2 \\ \mathbf{0} & \boldsymbol{A}_2 \end{bmatrix} \right) \boldsymbol{\psi}_{c,j} = \mathbf{0} \ , \quad j = 1,\ 2,\ \ldots\ ,\ np \tag{3.77}$$

The characteristic polynomial (3.76) can also be expressed as

$$det\left(\lambda \boldsymbol{I} - \boldsymbol{A}_1\right) det\left(\lambda \boldsymbol{I} - \boldsymbol{A}_2\right) = 0 \tag{3.78}$$

which, not surprisingly, implies that the eigenvalues of the combined system are exactly the eigenvalues of the structural system and of the shaping filter. If the eigenvector of (3.77) is partitioned in two, the *j*th eigenvalue problem can be formulated as

$$\left(\lambda_j \boldsymbol{I} - \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{B}_1\boldsymbol{H}_2 \\ \boldsymbol{0} & \boldsymbol{A}_2 \end{bmatrix} \right) \begin{bmatrix} \boldsymbol{\psi}_{c,j}^{(1)} \\ \boldsymbol{\psi}_{c,j}^{(2)} \end{bmatrix} = \boldsymbol{0} \ , \quad j = 1,\ 2,\ \dots\ ,\ np \tag{3.79}$$

or equivalently as

$$\left.\begin{aligned} &\left(\lambda_j \boldsymbol{I} - \boldsymbol{A}_1\right)\boldsymbol{\psi}_{c,j}^{(1)} + \boldsymbol{B}_1\boldsymbol{H}_2\boldsymbol{\psi}_{c,j}^{(2)} = \boldsymbol{0} \\ &\left(\lambda_j \boldsymbol{I} - \boldsymbol{A}_2\right)\boldsymbol{\psi}_{c,j}^{(2)} = \boldsymbol{0} \end{aligned}\right\} \quad j = 1,\ 2,\ \dots\ ,\ np \tag{3.80}$$

It is seen that if the eigenvalue λ_j originates from the shaping filter, the second equation of (3.80) is an eigenvalue problem with $\boldsymbol{\psi}_{c,j}^{(2)}$ as the corresponding eigenvector. From the first equation of (3.80), the remaining part of the eigenvector $\boldsymbol{\psi}_{c,j}$ can be determined as

$$\boldsymbol{\psi}_{c,j}^{(1)} = -\left(\lambda_j \boldsymbol{I} - \boldsymbol{A}_1\right)^{-1}\boldsymbol{B}_1\boldsymbol{H}_2\boldsymbol{\psi}_{c,j}^{(2)} \ , \quad j = 1,\ 2,\ \dots\ ,\ np \tag{3.81}$$

The observable part of the eigenvector, which is the "mode shape" of the shaping filter, is then calculated by

$$\boldsymbol{\Phi}_j = \begin{bmatrix} \boldsymbol{H}_1 & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\psi}_{c,j}^{(1)} \\ \boldsymbol{\psi}_{c,j}^{(2)} \end{bmatrix} = -\boldsymbol{H}_1\left(\lambda_j \boldsymbol{I} - \boldsymbol{A}_1\right)^{-1}\boldsymbol{B}_1\boldsymbol{H}_2\boldsymbol{\psi}_{c,j}^{(2)} \ , \quad j = 1,\ 2,\ \dots\ ,\ np \tag{3.82}$$

From (3.81) and (3.82) it is seen that the vector $\boldsymbol{\Phi}_j$ corresponding to an eigenvalue of the shaping filter, has changed due to the convolution process. It should also be noticed from (3.81) that the eigenvalues of the structural system must be different from the eigenvalues of the shaping filter, otherwise the matrix $(\lambda_j\boldsymbol{I} - \boldsymbol{A}_1)$ will lose rank and become impossible to invert. However, this does not create any problems here, since all eigenvalue are assumed distinct. Now, assume that the eigenvalue λ_j originates from the structural system. In this case the matrix $(\lambda_j\boldsymbol{I} - \boldsymbol{A}_2)$ in (3.80) will be non-singular and positive definite. Therefore, the only way that the second equation in (3.80) can be satisfied is if and only if $\psi_{c,j}^{(2)}$ is zero, which implies that the first equation in (3.80) is given by

$$\left(\lambda_j \boldsymbol{I} - \boldsymbol{A}_1\right)\psi_{c,j}^{(1)} = \boldsymbol{0} \ , \quad j = 1,\ 2,\ \dots\ ,\ np \tag{3.83}$$

This equation is a standard first-order eigenvalue problem for the structural system, with $\psi_{c,j}^{(1)}$ being a non-trivial eigenvector. Since this eigenvalue problem relates to the structural system the observable part of the eigenvector will indeed be a true mode shape. The associated scaled mode shape is therefore obtained from

$$\boldsymbol{\Phi}_j = \begin{bmatrix} \boldsymbol{H}_1 & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \psi_{c,j}^{(1)} \\ \boldsymbol{0} \end{bmatrix} = \boldsymbol{\Phi}_j \ , \quad j = 1,\ 2,\ \dots\ ,\ np \tag{3.84}$$

and it is seen to be equivalent to the original mode shape associated with the jth eigenvalue of the structural system.

So in conclusion:

☞ *The scaled mode shapes of the structural system are not affected by the convolution of the shaping filter of the excitation. The only possible scaling they may exhibit is individual and depends on what the output is according to* (3.70).

3.4.4 Example 3.3: Modal Decomposition

In section 3.3.2 the second-order white noise excited structural system was represented by the state space realization

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{F}\boldsymbol{x}(t) + \boldsymbol{B}w(t) \\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t) \end{aligned} \tag{3.85}$$

with the matrix triple $\{\boldsymbol{F}, \boldsymbol{B}, \boldsymbol{C}\}$ of the combined continuous-time system (3.46) is given by

$$\boldsymbol{F} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{3.86}$$

The eigenvalue problems can then be formulated as

$$\left(\begin{bmatrix} \lambda_j & 0 \\ 0 & \lambda_j \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \right) \begin{bmatrix} 1 \\ \lambda_j \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad j = 1,\ 2 \tag{3.87}$$

and the corresponding characteristic polynomial is

$$\lambda_j^2 + \frac{c}{m}\lambda_j + \frac{k}{m} = 0 \tag{3.88}$$

The solution of this polynomial leads to two complex conjugated eigenvalues given by

$$\lambda_1 = \frac{-c+\sqrt{c^2-4mk}}{2m} \quad , \quad \lambda_2 = \frac{-c-\sqrt{c^2-4mk}}{2m} \tag{3.89}$$

and the similarity transformation

$$\boldsymbol{F} = \frac{1}{\lambda_2 - \lambda_1}\begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix} \tag{3.90}$$

□

3.5 Summary

This chapter has dealt with the continuous-time mathematical description of structural systems. The concept of modal analysis has been introduced on a second-order differential equation system. It has been shown how the dynamic behaviour of such a system can be broken down into subproblems by a modal decomposition. Each of the subproblems are described by modal parameters, which can be divided into modal frequencies, modal damping, modal vectors, and modal scaling. Since the number of measurement channels in general is lower than the identified number of dynamic modes of a system, it is necessary to generalize the mathematical description of the structural system. Further, to use the ARMAV model as a discrete-time representation of an ambient excited structure, the ambient excitation is assumed to be the output of a linear shaping filter subjected to Gaussian white noise. Since the ARMAV model assumes a Gaussian white noise, the generalized structural system and the shaping filter have been combined into one mathematical model, which is excited by continuous-time Gaussian white noise. Further, to be able to extract other characteristics than the displacements of the observed mass point of the structure, this combined system has been generalized to make this possible. Through a modal decomposition of the combined system it has been shown that it is still possible to extract the modal frequencies, modal damping, and modal vectors of the structural system. However, the modal scaling will be affected by the shaping filter.