

SCALING FACTOR ESTIMATION BY THE MASS CHANGE METHOD

Manuel López Aenlle	Department of Construction and Manufacturing Engineering, University of Oviedo	Spain
Rune Brincker	Department of Building Technology and Structural Engineering, Aalborg University	Denmark
Alfonso Fernández Canteli	Department of Construction and Manufacturing Engineering, University of Oviedo	Spain
Luis M. Villa García	Department of Construction and Manufacturing Engineering, University of Oviedo	Spain

aenlle@uniovi.es

Abstract

When natural input modal analysis is performed, the acting forces are unknown; by this reason only un-scaled mode shapes may be obtained so that the FRF matrix can not be constructed. If the structure is modified and a new modal testing is carried out, the scaling factors can be determined using the modal parameters (natural frequencies and mode shapes) from both the modified and the unmodified structure. Mass change is in many cases the simplest way to perform structural modification, which involves repeated testing implying mass change in different points of the structure where the mode shapes are known. In this paper, several methods to estimate the scaling factors, based on the mass change method, are presented. The accuracy obtained through the methods proposed depends on the type of normalization used in the mode shapes, the mass change magnitude and the number and the location of the masses attached to the structure, which effect is also analyzed. Finally, it is shown how the scaling factors can be used to improve the modal updating procedures.

Nomenclature

Scaling factor	α	Number of modes	Nm
Stiffness matrix	$[k]$	Mass matrix	$[m]$
Damping matrix	$[c]$	Mass change matrix	$[\Delta m]$
Scaled mode shape	$\{\phi\}$	Un-scaled mode shape	$\{\psi\}$
Natural frequency of the unmodified structure		ω_0	
Scaled mode shape of the unmodified structure		$\{\phi\}_0$	
Un-scaled mode shape of the unmodified structure		$\{\psi\}_0$	
Natural frequency of the modified structure		ω_1	
Damping factor		ζ	
Scaled mode shape of the modified structure		$\{\phi\}_1$	
Un-scaled mode shape of the modified structure		$\{\psi\}_1$	

1 Introduction

Modal analysis has become a powerful technology which can be applied to a wider range of the structures. If the identified modal model is going to be used for structural response simulation, structural modification or health monitoring, the frequency response function (FRF) or the impulse response function (IRF) have to be known. To construct these matrices, the following information is needed for each mode:

- The natural frequency ω ,
- The damping factor ζ ,
- The mass normalized (scaled) mode shape $\{\phi\}$.

The scaling factor α and the un-scaled mode shape $\{\psi\}$ or the scaled mode shape $\{\phi\}$ can be used indistinctly.

Traditional modal analysis let us to estimate all the aforementioned parameters. On the contrary, when natural input modal analysis is performed, only the un-scaled mode shapes can be obtained for each mode. As a consequence an additional procedure to calculate the scaling factors α is needed.

If, as it is the usual case, the forces acting on a structure can not be measured, a way to estimate the scaling factors is to modify the dynamic behavior of the structure changing the stiffness or the mass and perform a new natural modal testing and analysis. The methods based on dynamic modification use both the modal parameters of the unmodified and modified structure. Therefore, the methods used to estimate the scaling factors are based on a more extensive experimental testing procedure consisting of the following stages:

1. Performing a first natural input modal testing and analysis to obtain the modal parameters of the unmodified structure ω_0 , $\{\psi_0\}$ and ζ_0 .
2. Modification of the structure (mass, stiffness or damping).
3. Performing a second natural input modal testing and analysis to obtain the modal parameters of the modified structure ω_1 , $\{\psi_1\}$ and ζ_1 .
4. Estimation of the scaling factors α_0 .

Recently, some algorithms have been proposed [1], [2], [5] and [6] to estimate the scaling factors based on attaching several masses in the points where the mode shapes are known. This family is here denoted as the mass change method.

In this paper, proportional damping is assumed, so that the modes will be real.

2 The scaling factor

The scaled and the un-scaled mode shape are related by the equation:

$$\{\phi\} = \alpha \cdot \{\psi\} \tag{1}$$

where α is the scaling factor that can be expressed as:

$$\alpha = \frac{1}{\sqrt{\{\psi\}^T \cdot [m] \cdot \{\psi\}}} \quad (2)$$

The magnitude of the scaling factor depends on the type of normalization used for $\{\psi\}$. If α_u and $\{\psi_u\}$ are the scaling factor and the mode shape when normalization to unity is used, the corresponding scaling factor using normalization to the length is given by:

$$\alpha_L = \alpha_u \sqrt{\{\psi_u\}^T \cdot \{\psi_u\}} \quad (3)$$

3 The mass change method

The mass change method consists on performing natural input modal analysis on both the original and the modified structure [2], [5], [6]. The modification is carried out attaching masses to the points of the structure where the mode shapes of the unmodified structure are known. The user selects the number, the magnitude and the location of the masses. The process is, schematically, shown in Figure 1.

In order to facilitate the mass modification and the calculation of the scaling factors, lumped masses are often used, so that the mass change matrix $[\Delta m]$ becomes, in general, diagonal.

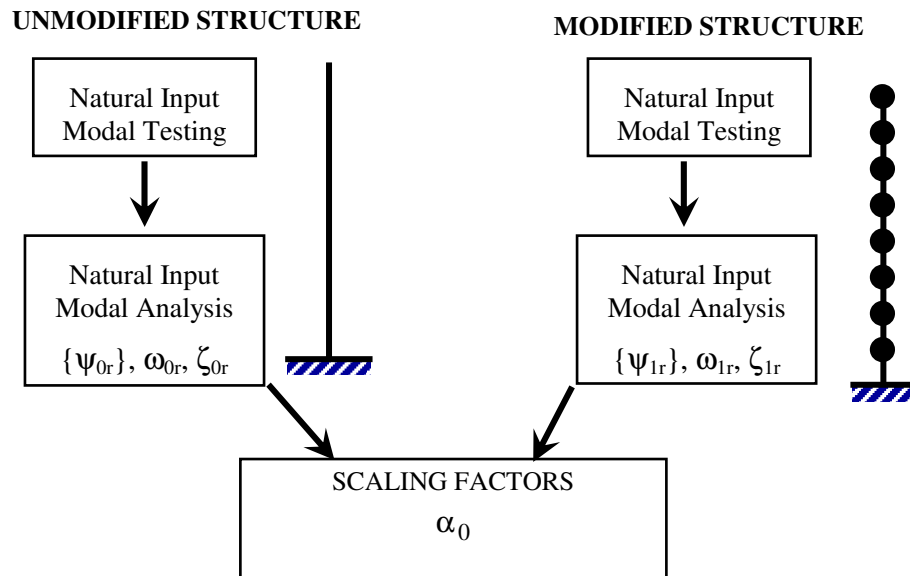


Figure 1. The mass change method.

The different methods proposed to estimate the scaling factors can be classified in two groups:

- Direct methods, which use the experimental modal parameters of both the modified and the unmodified structure. These methods can be derived from the eigenvalue equations of

both the modified and the unmodified structure or alternatively, from the sensitivity equations.

- Indirect methods, which use the updated analytical mass matrix or the analytical stiffness matrix, together with the modal parameters.

4 A simple method

A simple method can be derived from the eigenvalue equations of both the unmodified (original) and the modified structure [2]. The classical eigenvalue equation in case of no damping or proportional damping is:

$$[m] \cdot \{\phi_0\} \cdot \omega_0^2 = [k] \cdot \{\phi_0\}, \quad (4)$$

where $\{\phi_0\}$ is the mode shape, ω_0 the natural frequency, $[m]$ the mass matrix and $[k]$ the stiffness matrix. If we make a mass change so that the new mass matrix is $[m] + [\Delta m]$, then the eigenvalue equation becomes:

$$([m] + [\Delta m]) \cdot \{\phi_1\} \cdot \omega_1^2 = [k] \cdot \{\phi_1\}, \quad (5)$$

where $\{\phi_1\}$ and ω_1 are the new modal parameters of the modified problem.

If we now assume that the mass change is so small that the mode shapes does not change significantly, i.e., $\{\phi_1\} \cong \{\phi_0\} \cong \{\phi\}$ and subtracting Equations (4) and (5) we obtain:

$$[m] \cdot \{\phi\} (\omega_0^2 - \omega_1^2) - [\Delta m] \cdot \{\phi\} \cdot \omega_1^2 = 0 \quad (6)$$

Premultiplying Equation (6) by $\{\phi\}^T$ and taking into account the orthogonality of the modes, results in:

$$(\omega_0^2 - \omega_1^2) = \{\phi\}^T \cdot [\Delta m] \cdot \{\phi\} \cdot \omega_1^2. \quad (7)$$

Finally, combining Equation (1) and Equation (7) we obtain:

$$(\omega_0^2 - \omega_1^2) = \alpha^2 \cdot \{\psi\}^T \cdot [\Delta m] \cdot \{\psi\} \cdot \omega_1^2. \quad (8)$$

and the unknown scaling factor can be derived from [2] as:

$$\alpha = \sqrt{\frac{(\omega_0^2 - \omega_1^2)}{\omega_1^2 \cdot \{\psi\}^T \cdot [\Delta m] \cdot \{\psi\}}} \quad (9)$$

In Equation (9), both the modified and unmodified mode shapes can be used. Then, we have three possible alternatives [5] :

$$\alpha_{00} = \sqrt{\frac{(\omega_0^2 - \omega_1^2)}{\omega_1^2 \cdot \{\psi_0\}^T \cdot [\Delta m] \cdot \{\psi_0\}}}, \quad (10)$$

$$\alpha_{01} = \sqrt{\frac{(\omega_0^2 - \omega_1^2)}{\omega_1^2 \cdot \{\psi_0\}^T \cdot [\Delta m] \cdot \{\psi_1\}}}, \quad (11)$$

$$\alpha_{11} = \sqrt{\frac{(\omega_0^2 - \omega_1^2)}{\omega_1^2 \cdot \{\psi_1\}^T \cdot [\Delta m] \cdot \{\psi_1\}}}. \quad (12)$$

The results obtained from Equations (10) and (12) are independent of the type of normalization used whereas when applying Equation (11) normalization to the length should be used.

When considering Equation (10), with the mode shapes normalized to unity to estimate the scaling factors α_{00u} , the scaling factor α_{00L} normalized to the length is given by:

$$\alpha_{00L} = \alpha_{00u} \sqrt{\{\psi_{0u}\}^T \cdot \{\psi_{0u}\}}, \quad (13)$$

which is the exact relation, i.e., it is the same expression as Equation (3). The subscript 'u' indicates that mode shapes are normalized to unity

However, if Equation (11) is used to estimate the scaling factor α_{01u} , the scaling factor α_{01L} should be estimated using the expression:

$$\alpha_{01L} = \alpha_{01u} \sqrt{\{\psi_{0u}\}^T \cdot \{\psi_{1u}\}} \quad (14)$$

It turns out that the best results are obtained from Equation (11). However, the results can even be improved if the following relation are considered:

$$\alpha^* = \frac{\alpha_{00} + \alpha_{11}}{2}. \quad (15)$$

5 A coupled method

In the method proposed in the former paragraph, to estimate a scaling factor corresponding to a mode, only the mode shape and the natural frequency of this particular mode needs to be known. On the contrary, when coupled methods are applied, all mode shapes must be considered together, and all the scaling factors are estimated at the same time. In case of no damping or proportional damping, the equation of motion of a structure subjected to a force $\{p(t)\}$, i.e.:

$$[m] \cdot \{\ddot{u}\} + [c] \cdot \{\dot{u}\} + [k] \cdot \{u\} = \{p(t)\}, \quad (16)$$

provides the eigenvalue equation:

$$[m] \cdot \{\phi_0\} \cdot \omega_0^2 = [k] \cdot \{\phi_0\}. \quad (17)$$

If a mass change is performed resulting in a new mass matrix $[m]+[\Delta m]$, the new equation of motion becomes:

$$([m]+[\Delta m]) \cdot \{\ddot{u}\} + [c] \cdot \{\dot{u}\} + [k] \cdot \{u\} = \{p(t)\} \quad (18)$$

which provides the eigenvalue equation:

$$([m]+[\Delta m]) \cdot \{\phi_1\} \cdot \omega_1^2 = [k] \cdot \{\phi_1\}. \quad (19)$$

The mode shapes of the modified structure and the natural frequencies can be expressed, respectively, as, see [5] and [7]:

$$[\Phi_1] = [\Phi_0] \cdot [\Phi_b] \quad (20)$$

and

$$[\omega_1^2] = [\omega_b^2] \quad (21)$$

where the natural frequencies ω_b^2 and the mode shapes $\{\phi_b\}$ are obtained solving the eigenvalue equation, see [5] and [7]:

$$([I] + [\Phi_0]^T \cdot [\Delta m] \cdot [\Phi_0]) \cdot \{\phi_b\} \cdot \omega_b^2 = [\omega_0^2] \cdot \{\phi_b\}$$

The scaling factors for the modified structure α_0 and the unmodified structure α_1 can be estimated minimizing the equations, see [5]:

$$\begin{aligned} \{\varepsilon_\psi\} &= [\alpha_0 \cdot \Psi_0] \cdot [\Phi_b] - [\alpha_1 \cdot \Psi_1] \\ \{\varepsilon_\omega\} &= [\omega_1^2] - [\omega_b^2] \end{aligned} \quad (22)$$

with respect to $\{\alpha_0\}$ and $\{\alpha_1\}$. The modes $[\Phi_b]$ and natural frequencies $[\omega_b^2]$ are obtained solving the eigenvalue problem (19), whereas the natural frequencies and the mode shapes of both the modified and unmodified structure result from experimental modal analysis.

One advantage of this method is that the error between the estimated and the experimental mode shapes can be known from Equation (22). Furthermore, the scaling factors of both the modified and the unmodified mode shapes can be estimated.

6 Sensitivity equations

The sensitivity of the j degree of freedom of the i mode shape corresponding to a local change in the mass at the k degree of freedom is given by, see [4] and [6]:

$$\frac{\partial \{\phi_i\}}{\partial m_k} \cong -\frac{\phi_{ki}^2}{2} \{\phi_i\} + \phi_{ki} \sum_{t=1, t \neq i}^{Nm} \frac{\omega_i^2}{\omega_t^2 - \omega_i^2} \phi_{kt} \{\phi_t\} \quad (23)$$

whereas the sensitivity of the i pole to a local change in mass in the k DOF is given by, see [4] and [6]:

$$\frac{\partial \omega_i}{\partial m_k} \cong -\omega_i \frac{\phi_{ki}^2}{2} \quad (24)$$

If the mass change is performed simultaneously in several degree of freedoms, and finite difference approximation is used, the Equations (23) and (24) become:

$$\Delta\{\phi_i\} \cong -\{\phi_i\}^T [\Delta m] \cdot \{\phi_i\} \cdot \{\phi_i\} + \sum_{t=1, t \neq i}^{Nm} \frac{\omega_i^2}{\omega_t^2 - \omega_i^2} \{\phi_i\}^T [\Delta m] \cdot \{\phi_t\} \cdot \{\phi_t\} \quad (25)$$

$$\Delta \omega_i \cong -\frac{1}{2} \omega_i \cdot \alpha_i^2 \cdot \{\phi_i\}^T [\Delta m] \cdot \{\phi_i\}. \quad (26)$$

Substituting Equation (1) in Equations (25) and (26), leads to:

$$\Delta\{\psi_i\} \cong -\{\psi_i\}^T [\Delta m] \cdot \{\psi_i\} \cdot \alpha_i^2 \cdot \{\psi_i\} + \sum_{t=1, t \neq i}^{Nm} \frac{\omega_i^2}{\omega_t^2 - \omega_i^2} \cdot \{\psi_i\}^T [\Delta m] \cdot \{\psi_t\} \cdot \alpha_t^2 \cdot \{\psi_t\} \quad (27)$$

$$\Delta \omega_i \cong -\frac{1}{2} \omega_i \cdot \alpha_i^2 \cdot \{\psi_i\}^T [\Delta m] \cdot \{\psi_i\}, \quad (28)$$

Where by it has been assumed that the scaling factors do not vary significantly.

6.1 A simple method

From the sensitivity Equation (28) a new simple formula can be derived [6]:

$$\alpha = \sqrt{\frac{2\Delta\omega}{\omega_i \cdot \{\psi_i\}^T [\Delta m] \cdot \{\psi_i\}}} \quad (29)$$

which can also be derived from expression (9) using the approximation [2]:

$$\omega_0^2 - \omega_1^2 \cong 2\omega_1(\omega_0 - \omega_1) = 2\omega_1 \cdot \Delta\omega \quad (30)$$

so that it can be concluded that Equation (9) provides better results than Equation (29).

6.2 A coupled method

A method coupling all the modes considered, can also be derived from the sensitivity equations (27) and (28). For the mode i the following equations can be written:

$$\{\psi_{i1}\} - \{\psi_{i0}\} \cong -\{\psi_{i0}\}^T [\Delta m] \cdot \{\psi_{i0}\} \cdot \alpha_{i0}^2 \cdot \{\psi_{i0}\} + \sum_{t=1, t \neq i}^{Nm} \frac{\omega_{i0}^2}{\omega_{t0}^2 - \omega_{i0}^2} \cdot \{\psi_{i0}\}^T [\Delta m] \cdot \{\psi_{t0}\} \cdot \alpha_{t0}^2 \cdot \{\psi_{t0}\} \quad (31)$$

$$\omega_{i1} - \omega_{i0} = -\frac{1}{2} \omega_{i0} \cdot \alpha_{i0}^2 \cdot \{\psi_{i0}\}^T [\Delta m] \cdot \{\psi_{i0}\}. \quad (32)$$

The scaling factors can be estimated by minimizing Equations (31) and (32) with respect to $\alpha_{10}, \alpha_{20}, \dots, \alpha_{i0}, \dots, \alpha_{Nm0}$.

7 Comparison of the methods

The accuracy obtained with the methods described in the preceding paragraphs depends, generally, on the mode, the mass change magnitude, the number of modes considered in the analysis, the location and number of the masses, etc. However, general rules can be provided.

The simple formula proposed in paragraph 5 is easy to apply and to estimate a scaling factor, only the mode shape and the natural frequency of this particular mode has to be known. The best results are obtained when using Equations (11) and (15) and normalization to the length should be considered.

To compare the accuracy obtained with the described methods, one thousand simulations were performed on a simple dynamic system with 15 degree of freedoms:

$$[k] = k \begin{bmatrix} 2 & -1 & . & 0 & 0 \\ -1 & 2 & . & 0 & 0 \\ . & . & . & . & . \\ 0 & 0 & . & 2 & -1 \\ 0 & 0 & . & -1 & 2 \end{bmatrix} \quad [m] = m \begin{bmatrix} 1 & . & 0 & 0 \\ 0 & 1 & . & 0 \\ . & . & . & . \\ 0 & 0 & . & 1 \\ 0 & 0 & . & 1 \end{bmatrix} \quad (33)$$

Figure 2a shows the errors obtained from Equations (11) and (15) when 12 modes are considered and the masses attached in 5 points located in random positions. As can be seen, both equations provide approximately the same accuracy in the first modes but Equation (15) provides clearly better results for the higher modes. In Figure 2b, the standard deviation of the values obtained from simulations is represented.

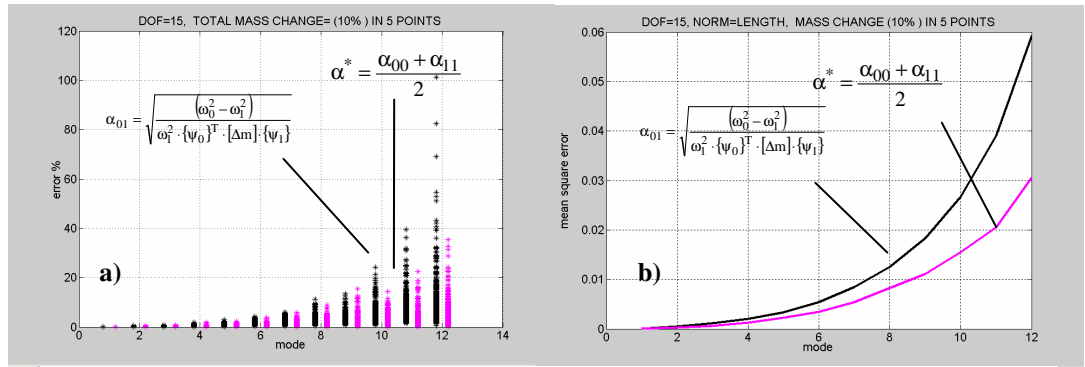


Figure 2. Comparison between Equations (11) and (15).

With coupled methods, all modes are taken together what implies more complicated algorithms (optimization algorithms) to determine the scaling factors. The results are independent of the type

of normalization. The accuracy obtained with this method is highly dependent on the modes considered in the estimation so that the accuracy improves as more modes are considered.

The simple formula provides best results when only few modes are considered in the estimation. The coupled methods should only be preferred when the number of modes are near to the number of observation points. In Figure 3a a comparison between the simple formula and the coupled method is presented, corresponding to a 15 degree of freedom system. In Figure 3a the mean square error of the scaling factors, obtained from simulations when 5 modes are considered, is presented, whereas Figure 3b shows the results when 13 modes are considered. As can be seen, the simple formula provides good results independently of the number of modes considered, whereas the coupled method is only recommended when a high number of modes are implied.

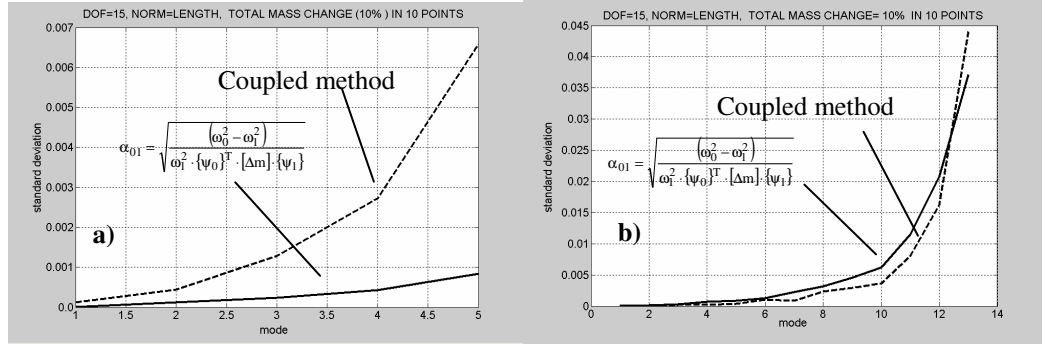


Figure 3. Comparison between the simple formula and the coupled method. a) Standard deviation with 5 modes. b) Standard deviation with 13 modes.

8 Mass change strategy

It can be demonstrated that to reduce the uncertainty in the scaling factor estimation we have to minimize the inaccuracies of the estimates in the modal testing analysis stage as well as the difference between the modified and unmodified mode shapes, [2][6].

The difference between the original and the modified mode shapes is minimized when:

- More masses are attached to the structure.
- The masses are well distributed.
- The masses are located in optimal positions (peaks and valleys of the mode shapes)
- The magnitude of the mass change is small.

Nevertheless, a minimum change of the mass magnitude is required [2] to guarantee a minimum frequency shift because the uncertainty in the modal analysis identification. On the other hand, the mass change should not be too high in order to minimize the difference between the modified and the unmodified mode shapes.

A mass change of 5% of the total mass, see [6], could be a reasonable mass change.

9 The mass change method and modal updating

In natural input modal analysis, the mass change method is suitable to determine the scaling factors, which involves additional experimental testing. On one hand, the main disadvantage of the mass change method is that several masses have to be attached to the structure, which occasionally is difficult and expensive. On the other hand, we obtain additional information which can be used to improve the accuracy of the modal updating process.

If the mass change method is used by performing only one mass change, the following information can be obtained after the corresponding natural input modal testing and analysis:

- The natural frequencies of both the modified and the unmodified structure:

$$\left[\omega_0^2 \right]^x \text{ and } \left[\omega_1^2 \right]^x$$

where the superscript 'x' denotes experimental parameters

- The mode shapes of both the modified and the unmodified structure:

$$\left[\{\psi_0\} \right]^x \text{ and } \left[\{\psi_1\} \right]^x$$

The scaling factors $\{\alpha_0\}$ of the unmodified mode shapes can be estimated applying the methods described in the preceding paragraphs. The natural frequencies and the mode shapes can be estimated with reasonable accuracy making use of natural input modal analysis, see [2]

The accuracy obtained in the scaling factor estimation depends on the mode, the number of modes considered in the analysis, the number of attached masses, the location of the masses, the error in the natural frequency and mode shape estimation and the method involved to estimate the scaling factors. However, the scaling factors corresponding to the first two or three modes can always be estimated with a good accuracy using the methods described in the previous paragraphs. The scaling factors are very useful information in modal updating but only the first ones should be used.

The analytical mass $[m]^a$ and stiffness matrices $[k]^a$ can be optimized by fulfilling the following equations:

- The eigenvalue equations of both the modified and the unmodified structure:

$$\left[[k]^a - [m]^a \omega_{0i}^2 \right] \{\psi_{0i}\}^x = \{0\} \quad (i = 1, \dots, n \text{ modes}) \quad (34)$$

$$\left[[k]^a - ([m]^a + [\Delta m]) \omega_{1i}^2 \right] \{\psi_{1i}\}^x = \{0\} \quad (i = 1, \dots, n \text{ modes})$$

- The orthogonality conditions of both the modified and the unmodified structure:

$$\begin{aligned} \{\psi_{0i}\}^x{}^T \cdot [k]^a \cdot \{\psi_{0j}\}^x &= 0 \quad (i \neq j) \\ \{\psi_{1i}\}^x{}^T \cdot [k]^a \cdot \{\psi_{1j}\}^x &= 0 \quad (i \neq j) \\ \{\psi_{0i}\}^x{}^T \cdot [m]^a \cdot \{\psi_{0j}\}^x &= 0 \quad (i \neq j) \\ \{\psi_{1i}\}^x{}^T \cdot ([m]^a + [\Delta m]) \cdot \{\psi_{1j}\}^x &= 0 \quad (i \neq j) \end{aligned} \quad (35)$$

- The first scaling factors:

$$\begin{aligned} \{\psi_i\}^{x^T} \cdot [k]^a \cdot \{\psi_i\}^x &= \frac{\omega_i^{x^2}}{\alpha_i^2} & (i = 1, 2) \\ \{\psi_i\}^{x^T} \cdot [m]^a \cdot \{\psi_i\}^x &= \frac{1}{\alpha_i^2} & (i = 1, 2) \end{aligned} \quad (36)$$

The accuracy of the modal updating process can be improved if more experimental information can be obtained performing additional mass changes. If the magnitude or the location of several masses is modified, additional information (natural frequencies and mode shapes) can be derived from a new modal testing and analysis.

This new mass modification is, in general, easy to perform in small structures, because the magnitude of the masses is small. In big structures, a mass modification becomes relatively simple if we are working with masses that are easy to move [6].

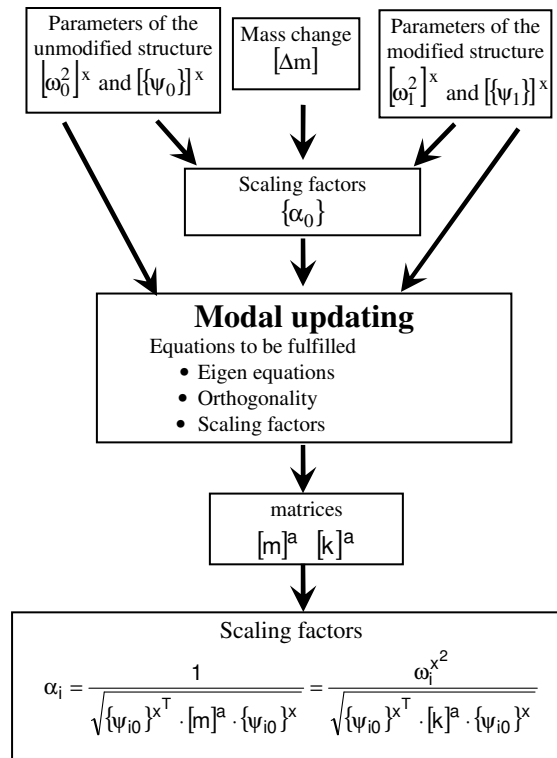


Figure 4. The mass change method and modal updating.

At the end of the modal updating process we get the analytical mass and stiffness matrices. An important advantage is that if we have a good estimation of these matrices, we can estimate the scaling factors using the analytical mass matrix:

$$\alpha_i = \frac{1}{\sqrt{\{\psi_i\}^x \cdot [m]^a \cdot \{\psi_i\}^x}} \quad (37)$$

Or the stiffness matrix

$$\alpha_i = \frac{\omega_i^{x^2}}{\sqrt{\{\psi_i\}^x \cdot [k]^a \cdot \{\psi_i\}^x}} \quad (38)$$

The process is shown in Figure 4.

10 Conclusions

In this paper some methods are described to determine the scaling factors in natural input modal analysis, by modifying the mass in several points of the structure.

The simple method provides the best results when only few modes are considered in the estimation whereas the coupled method are recommended when the number of modes is close to the number of observation points.

The combination of the scaling factors and modal updating in an iterative process allow us to improve the estimation of the scaling factors as well as the analytical mass and stiffness matrix.

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