Efficient uncertainty computation for modal parameters in stochastic subspace identification

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Abstract
Stochastic Subspace Identification methods have been extensively used for the modal analysis of mechanical, civil or aeronautical structures for the last ten years to estimate modal parameters. Recently an uncertainty computation scheme has been derived allowing the computation of uncertainty bounds for modal parameters. However, this scheme is numerically feasible only for quite low model orders, as the computations involve big matrix products that are memory and computationally taxing. In this paper, a fast computation scheme is presented that reduces the computational burden of the uncertainty computation scheme significantly. The fast computation is illustrated on the modal analysis of two bridges.

1 Introduction

Subspace-based system identification methods have proven to be efficient for the identification of linear time-invariant systems (LTI), fitting a linear model to input/output or output only measurements taken from a system. An overview of subspace methods can be found in [3, 4, 11, 20].

During the last decade, subspace methods found a special interest in mechanical, civil and aeronautical engineering for modal analysis, namely the identification of vibration modes of structures from the eigenvalues (natural frequencies and damping ratios) and observed eigenvectors (mode shapes) of a LTI system. For Operational Modal Analysis, the identification of a structure under operation conditions, it is often impractical to excite the structure artificially, so that vibration measurements are taken under unmeasured ambient excitation. Therefore, identifying an LTI system from output-only measurements is a basic service in vibration analysis, see e.g. [1, 17].

For any system identification method, the estimated modal parameters are afflicted with statistical uncertainty for many reasons, e.g. finite number of data samples, undefined measurement noises, non-stationary excitations, nonlinear structure, model order reduction, etc. Then the system identification algorithms do not yield the exact system matrices and identification results are subject to variance errors. For many system identification methods, the estimated parameters are asymptotically normal distributed, e.g. for estimates from prediction error methods [15], maximum likelihood methods [18], or the here considered subspace methods [2, 6]. A detailed formulation of the covariance computation for the modal parameters from covariance-driven stochastic subspace identification is given in [19], where covariance estimates are based on the propagation of first-order perturbations from the data to the modal parameters. These methods are very attractive for modal analysis, as covariance estimates are obtained in one shot: From the same data set that is used to estimate the modal parameters, all the covariance information is obtained by cutting the available data into blocks, which is then propagated to the modal parameters, without the need of computing the modal parameters on the blocks.

The covariance computation of modal parameters from subspace identification in [19] is computationally taxing. In this paper, an efficient computation scheme is presented, where the sensitivity computation from [19] for the uncertainty propagation is reformulated.

This paper is organized as follows. In Section 2, the covariance-driven stochastic subspace identification is given. In Section 3, the principle of the covariance computations is explained. In Section 4, notations and results of the uncertainty computations obtained in [19] are recalled. In Section 5, a new algorithm from [10] is sketched that provides exactly the same uncertainty bounds as in Section 4 but at a lower computational cost. The efficiency of the new algorithms is demonstrated in Section 6 for the computation of the modal parameters and their covariance of two bridges.

2 Stochastic Subspace Identification (SSI)

2.1 Models and parameters

The behavior of a vibrating structure is described by a continuous-time, time-invariant, linear dynamical system, modeled by the vector differential system

$$\mathcal{M}\ddot{x}(t) + \mathcal{C}\dot{x}(t) + \mathcal{K}x(t) = v(t) \tag{1}$$

where t denotes continuous time; $\mathcal{M}, \mathcal{C}, \mathcal{K} \in \mathbb{R}^{d\times d}$ are mass, damping, and stiffness matrices, respectively; the (high dimensional) state vector $x(t)$ is the displacement vector of the d degrees of freedom of the structure; the external unmeasured force $v(t)$ is unmeasured noise. Observing model (1) at r sensor locations (e.g. by acceleration, velocity or displacement measurements) and sampling it at some rate $1/\tau$ yields the discrete model in state-space form

$$\begin{cases} x_{k+1} = Ax_k + v_k \\ y_k = Cx_k + w_k, \end{cases} \tag{2}$$

with the states $x_k = \begin{bmatrix} x(k\tau)^T & \dot{x}(k\tau)^T \end{bmatrix}^T \in \mathbb{R}^n$, the outputs $y_k = y(k\tau) \in \mathbb{R}^r$ and the unobserved input and output disturbances v and w. The matrices $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{r\times n}$ are the state transition and observation matrices, respectively. Define r_0 as the number of so-called *projection channels* or *reference sensors* with $r_0 \leq r$, which are a subset of the r sensors and can be used for reducing the size of the matrices in the identification process [17].

The eigenstructure (λ, φ) of system (2) is defined by the eigenvalues and eigenvectors of A and by C:

$$(A - \lambda_i I)\phi_i = 0, \quad \varphi_i = C\phi_i \tag{3}$$

The modal parameters of system (1) are equivalently found in the eigenstructure (λ, φ) of (2). The modal frequencies f_i and damping coefficients ρ_i are recovered directly from the eigenvalues λ_i by

$$f_i = \frac{a_i}{2\pi\tau}, \quad \rho_i = \frac{100|b_i|}{\sqrt{a_i^2 + b_i^2}}, \tag{4}$$

where $a_i = |\arctan \Im(\lambda_i)/\Re(\lambda_i)|$ and $b_i = \ln|\lambda_i|$.

2.2 Covariance-driven subspace identification

Let the parameters p and q be given such that $pr \geq qr_0 \geq n$. A matrix $\mathcal{H}_{p+1,q} \in \mathbb{R}^{(p+1)r\times qr_0}$ is built from the output data according to the chosen subspace algorithm, which will be called *subspace matrix* in the following. For *covariance-driven* SSI [3, 4, 17], let $R_i \stackrel{\text{def}}{=} \mathbf{E}(y_k y_{k-i}^{(\text{ref})T}) \in \mathbb{R}^{r \times r_0}$ be the theoretical output-correlations and the block Hankel matrix

$$\mathcal{H}_{p+1,q} \stackrel{\text{def}}{=} \begin{bmatrix} R_1 & R_2 & \dots & R_q \\ R_2 & R_3 & \dots & R_{q+1} \\ \vdots & \vdots & \ddots & \vdots \\ R_{p+1} & R_{p+2} & \dots & R_{p+q} \end{bmatrix} \stackrel{\text{def}}{=} \text{Hank}(R_i) \tag{5}$$

the subspace matrix, where $\mathbf{E}$ denotes the expectation operator. It enjoys the factorization property

$$\mathcal{H}_{p+1,q} = \mathcal{O}_{p+1} \mathcal{Z}_q \tag{6}$$

into the observability matrix

$$\mathcal{O}_{p+1} \stackrel{\text{def}}{=} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^p \end{bmatrix}$$

and the stochastic controllability matrix $\mathcal{Z}_q$. For simplicity, skip the subscripts of $\mathcal{H}_{p+1,q}$, $\mathcal{O}_{p+1}$ and $\mathcal{Z}_q$ in the following.

The observation matrix C is then found in the first block-row of the observability matrix $\mathcal{O}$. The state transition matrix A is obtained from the shift invariance property of $\mathcal{O}$, namely as the least squares solution of

$$\mathcal{O}^{\uparrow} A = \mathcal{O}^{\downarrow}, \text{ where } \mathcal{O}^{\uparrow} \stackrel{\text{def}}{=} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{p-1} \end{bmatrix}, \; \mathcal{O}^{\downarrow} \stackrel{\text{def}}{=} \begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^p \end{bmatrix}. \tag{7}$$

The least squares solution can be obtained from

$$A = \mathcal{O}^{\uparrow\dagger} \mathcal{O}^{\downarrow}, \tag{8}$$

where † denotes the Moore-Penrose pseudoinverse. The modal parameters are finally recovered from (3)–(4).

Using an actual data set $(y_k)_{k=1,\dots,N+p+q}$, the output correlations

$$\widehat{R}_i \stackrel{\text{def}}{=} \frac{1}{N} \sum_{k=1}^{N} y_k y_{k-i}^{(\text{ref})T}$$

are estimated and the empirical subspace matrix $\widehat{\mathcal{H}} = \text{Hank}(\widehat{R}_i)$ computed. The observability estimate $\widehat{\mathcal{O}}$ is obtained from a thin SVD of the matrix $\widehat{\mathcal{H}}$ and its truncation at the desired model order n:

$$\begin{aligned} \widehat{\mathcal{H}} &= U \Sigma V^T \\ &= \begin{bmatrix} U_1 & U_0 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_0^T \end{bmatrix}, \end{aligned} \tag{9}$$

$$\widehat{\mathcal{O}} = U_1 \Sigma_1^{1/2}, \tag{10}$$

where

$$U_1 = \begin{bmatrix} u_1 & \dots & u_n \end{bmatrix} \in \mathbb{R}^{(p+1)r \times n}, \; V_1 = \begin{bmatrix} v_1 & \dots & v_n \end{bmatrix} \in \mathbb{R}^{q r_0 \times n}, \; \Sigma_1 = \text{diag}\{\sigma_1, \dots, \sigma_n\} \in \mathbb{R}^{n \times n}.$$

Finally, the system matrix estimates $(\widehat{A}, \widehat{C})$ and estimates of the modal parameters are obtained as outlined above.

3 Preliminaries for covariance computations

In this section, the notations and basic principles of the covariance computations in the subsequent sections are introduced. In particular, the concept of perturbation to compute uncertainty bounds is defined. Furthermore, the computation of the covariance of the estimated subspace matrix $\widehat{\mathcal{H}}$ is explained and related to uncertainty propagation to other variables.

3.1 Definitions

First, the notation of uncertainties is defined. Let X be a smooth and bounded matrix function of an artificial scalar variable, where $X(0)$ is the "true value" of X and $X(t)$ is a perturbed (estimated) value of X, where t is small. Thus, the matrix can be expressed as $X = X(t)$ when computed from an estimated X, while $X(0)$ is their true but unknown value. Using the Taylor expansion

$$X(t) = X(0) + t\dot{X}(0) + O(t^2),$$

a first-order *perturbation* is denoted by $\Delta X \stackrel{\text{def}}{=} t\dot{X}(0) = X(t) - X(0) + O(t^2)$ for small t. Then, a perturbation on X can be propagated for any function $Y = f(X)$ by a first-order Taylor series of $f(X(t))$, yielding $\operatorname{vec}(\Delta Y) = \mathcal{J}_{Y,X} \operatorname{vec}(\Delta X)$, where $\mathcal{J}_{Y,X}$ is the *sensitivity* of $\operatorname{vec}(Y)$ with respect to $\operatorname{vec}(X)$ and where vec denotes the vectorization operator, which stacks the columns of a matrix on top of each other. Thus, from applying first-order perturbations the sensitivity matrix $\mathcal{J}_{Y,X}$ is obtained from a relation between $\operatorname{vec}(\Delta Y)$ and $\operatorname{vec}(\Delta X)$.

If $\operatorname{vec}(X)$ is an estimate that is asymptotically normal distributed, so is $\operatorname{vec}(Y)$ and the covariance of $\operatorname{vec}(Y)$ and the covariance of $\operatorname{vec}(X)$ satisfy asymptotically the relation

$$\operatorname{cov}(\operatorname{vec}(Y)) = \mathcal{J}_{Y,X} \operatorname{cov}(\operatorname{vec}(X))\, \mathcal{J}_{Y,X}^T. \tag{11}$$

The matrix $\Sigma_{\mathcal{H}} = \operatorname{cov}(\operatorname{vec}\widehat{\mathcal{H}})$ is denoted as the *covariance of the subspace matrix*. It can be obtained from cutting the available sensor data into blocks and is explained in Section 3.2. It is the starting point in the uncertainty propagation in (11) and it is the objective to compute the sensitivity matrices of the modal parameters with respect to the vectorized subspace matrix.

The following notation is used throughout the following sections. Let I_a be the $a \times a$ identity matrix and $0_{a\times b}$ the $a \times b$ zero matrix. Define the selection matrices

$$S_1 \stackrel{\text{def}}{=} \begin{bmatrix} I_{pr} & 0_{r\times pr} \end{bmatrix}, \ S_2 \stackrel{\text{def}}{=} \begin{bmatrix} 0_{r\times pr} & I_{pr} \end{bmatrix},$$

such that $S_1\mathcal{O} = \mathcal{O}^\uparrow$, $S_2\mathcal{O} = \mathcal{O}^\downarrow$. Furthermore, define the permutation matrix

$$\mathcal{P}_{a,b} \stackrel{\text{def}}{=} \sum_{k=1}^{a} \sum_{l=1}^{b} E_{k,l}^{a,b} \otimes E_{l,k}^{b,a}, \tag{12}$$

where $E_{k,l}^{a,b} \in \mathbb{R}^{a\times b}$ are matrices which are equal to 1 at entry (k,l) and zero elsewhere, and $\otimes$ denotes the Kronecker product [5]. In this context, the relation $\operatorname{vec}(QRS) = (S^T \otimes R)\operatorname{vec}(Q)$ is used for compatible matrices Q, R and S.

3.2 Estimating the covariance of the subspace matrix

For an estimation of the covariance $\Sigma_{\mathcal{H}}$, the available data is separated into n_b blocks of the same length N_b for simplicity, with $n_b \cdot N_b = N$. Each block may be long enough to assume statistical independence between the blocks. The correlations

$$\widehat{R}_i^{(j)} \stackrel{\text{def}}{=} \frac{1}{N_b} \sum_{k=1+(j-1)N_b}^{jN_b} y_k y_{k-i}^{(\text{ref})T}$$

are computed each of the blocks and the corresponding Hankel matrix $\widehat{\mathcal{H}}_j = \text{Hank}(\widehat{R}_i^{(j)})$ is filled. Then, $\widehat{\mathcal{H}} = \frac{1}{n_b}\sum_{j=1}^{n_b} \widehat{\mathcal{H}}_j$ and the covariance of the subspace matrix follows from the covariance of the sample mean as

$$\Sigma_{\mathcal{H}} = \frac{1}{n_b(n_b-1)} \sum_{j=1}^{n_b} \left(\text{vec}(\widehat{\mathcal{H}}_j) - \text{vec}(\widehat{\mathcal{H}})\right)\left(\text{vec}(\widehat{\mathcal{H}}_j) - \text{vec}(\widehat{\mathcal{H}})\right)^T. \tag{13}$$

4 Sensitivity computation from [19]

In this section, the sensitivities of the system matrices and the modal parameters with respect to the subspace matrix are recalled from [19].

4.1 Sensitivity computation of the system matrices A and C

The sensitivity computation of the matrices A and C is done in two steps: First, a perturbation $\Delta\mathcal{H}$ of the subspace matrix is propagated to a perturbation $\Delta\mathcal{O}$ of the observability matrix, and second, a perturbation $\Delta\mathcal{O}$ is propagated to perturbations ΔA and ΔC in the system matrices.

In order to obtain $\Delta\mathcal{O}$, the sensitivities of the singular values and vectors in (9) are necessary. They have been derived in [18].

Lemma 1 ([18]). *Let σ_i, u_i and v_i be the ith singular value, left and right singular vector of the matrix $\mathcal{H} \in \mathbb{R}^{(p+1)r\times qr_0}$ in* (9) *and $\Delta\mathcal{H}$ a small perturbation on $\mathcal{H}$. Then it holds*

$$\Delta\sigma_i = (v_i \otimes u_i)^T \text{vec}(\Delta\mathcal{H}),\ B_i \begin{bmatrix}\Delta u_i \\ \Delta v_i\end{bmatrix} = C_i \text{vec}(\Delta\mathcal{H}),$$

where

$$B_i \stackrel{\text{def}}{=} \begin{bmatrix} I_{(p+1)r} & -\frac{1}{\sigma_i}\mathcal{H} \\ -\frac{1}{\sigma_i}\mathcal{H}^T & I_{qr_0}\end{bmatrix},\quad C_i \stackrel{\text{def}}{=} \frac{1}{\sigma_i}\begin{bmatrix} v_i^T \otimes (I_{(p+1)r} - u_i u_i^T) \\ (u_i^T \otimes (I_{qr_0} - v_i v_i^T))\mathcal{P}_{(p+1)r,qr_0}\end{bmatrix}. \tag{14}$$

Using this result, the sensitivity of the observability matrix is derived in [19].

Lemma 2 ([19]). *Let B_i and C_i be given in Lemma 1 for $i = 1,\ldots,n$. Then,*

$$\text{vec}(\Delta\mathcal{O}) = \mathcal{J}_{\mathcal{O},\mathcal{H}} \text{vec}(\Delta\mathcal{H})$$

where $\mathcal{J}_{\mathcal{O},\mathcal{H}} \in \mathbb{R}^{(p+1)rn\times(p+1)rqr_0}$ with

$$\mathcal{J}_{\mathcal{O},\mathcal{H}} \stackrel{\text{def}}{=} \frac{1}{2}\left(I_n \otimes U_1\Sigma_1^{-1/2}\right) S_4 \begin{bmatrix}(v_1\otimes u_1)^T \\ \vdots \\ (v_n \otimes u_n)^T\end{bmatrix} + \left(\Sigma_1^{1/2} \otimes \begin{bmatrix} I_{(p+1)r} & 0_{(p+1)r\times qr_0}\end{bmatrix}\right)\begin{bmatrix} B_1^\dagger C_1 \\ \vdots \\ B_n^\dagger C_n\end{bmatrix},$$

$$S_4 \stackrel{\text{def}}{=} \sum_{k=1}^{n} E_{(k-1)n+k,k}^{n^2,n}, \tag{15}$$

Proof. From the definition of $\mathcal{O}$ follows

$$\Delta\mathcal{O} = U_1\Delta\Sigma_1^{1/2} + \Delta U_1\Sigma_1^{1/2} = U_1\frac{1}{2}\Sigma_1^{-1/2}\Delta\Sigma_1 + \Delta U_1\Sigma_1^{1/2},$$
$$\text{vec}(\Delta\mathcal{O}) = \frac{1}{2}\left(I_n \otimes U_1\Sigma_1^{-1/2}\right)\text{vec}(\Delta\Sigma_1) + \left(\Sigma_1^{1/2} \otimes I_{(p+1)r}\right)\text{vec}(\Delta U_1).$$

The sensitivities of Σ_1 and U_1 are obtained from Lemma 1 by stacking as

$$\begin{bmatrix}\Delta\sigma_1\\ \vdots\\ \Delta\sigma_n\end{bmatrix} = \begin{bmatrix}(v_1\otimes u_1)^T\\ \vdots\\ (v_n\otimes u_n)^T\end{bmatrix}\text{vec}(\Delta\mathcal{H}), \quad \text{vec}\begin{bmatrix}\Delta U_1\\ \Delta V_1\end{bmatrix} = \begin{bmatrix}B_1^\dagger C_1\\ \vdots\\ B_n^\dagger C_n\end{bmatrix}\text{vec}(\Delta\mathcal{H}),$$

and the assertion follows. Note that $\mathcal{J}_{\mathcal{O},\mathcal{H}} = \mathcal{B} + \mathcal{C}$ in [19]. □

The results of [19] on the sensitivity of the system matrices are collected in the following lemma.

Lemma 3 ([19])**.** *Let the system matrix A be obtained from $\mathcal{O}$ in* (8) *and C from the first block row of $\mathcal{O}$. Then, a perturbation in $\mathcal{O}$ is propagated to A and C by*

$$\text{vec}(\Delta A) = \mathcal{J}_{A,\mathcal{O}}\,\text{vec}(\Delta\mathcal{O}), \quad \text{vec}(\Delta C) = \mathcal{J}_{C,\mathcal{O}}\,\text{vec}(\Delta\mathcal{O}),$$

where $\mathcal{J}_{A,\mathcal{O}} \in \mathbb{R}^{n^2\times(p+1)rn}$, $\mathcal{J}_{C,\mathcal{O}} \in \mathbb{R}^{rn\times(p+1)rn}$, with

$$\begin{aligned}\mathcal{J}_{A,\mathcal{O}} &\stackrel{\text{def}}{=} (I_n \otimes \mathcal{O}^{\uparrow\dagger}S_2) - (A^T \otimes \mathcal{O}^{\uparrow\dagger}S_1) + ((\mathcal{O}^{\downarrow T}S_2 - A^T\mathcal{O}^{\uparrow T}S_2) \otimes (\mathcal{O}^{\uparrow T}\mathcal{O}^{\uparrow})^{-1})\mathcal{P}_{(p+1)r,n},\\ \mathcal{J}_{C,\mathcal{O}} &\stackrel{\text{def}}{=} I_n \otimes \begin{bmatrix}I_r & 0_{r,pr}\end{bmatrix}.\end{aligned} \tag{16}$$

Proof. Using the product rule for the sensitivity of $A = \mathcal{O}^{\uparrow\dagger}\mathcal{O}^{\downarrow} = (\mathcal{O}^{\uparrow T}\mathcal{O}^{\uparrow})^{-1}\mathcal{O}^{\uparrow T}\mathcal{O}^{\downarrow}$ and Kronecker algebra leads to the assertion. Note that $\mathcal{J}_{A,\mathcal{O}} = \mathcal{A}_1$ and $\mathcal{J}_{C,\mathcal{O}} = \mathcal{A}_2$ in [19]. □

Then, the covariances of the vectorized system matrices A and C satisfy

$$\Sigma_{A,C} \stackrel{\text{def}}{=} \text{cov}\left(\begin{bmatrix}\text{vec}(A)\\ \text{vec}(C)\end{bmatrix}\right) = \begin{bmatrix}\mathcal{J}_{A,\mathcal{O}}\\ \mathcal{J}_{C,\mathcal{O}}\end{bmatrix}\mathcal{J}_{\mathcal{O},\mathcal{H}}\,\Sigma_{\mathcal{H}}\,\mathcal{J}_{\mathcal{O},\mathcal{H}}^T\begin{bmatrix}\mathcal{J}_{A,\mathcal{O}}\\ \mathcal{J}_{C,\mathcal{O}}\end{bmatrix}^T. \tag{17}$$

4.2 Sensitivity computation of the modal parameters

In [19], the sensitivity derivations for the eigenvalues and eigenvectors of a matrix and subsequently for the modal parameters are stated, based on derivations in [14, 18]. They are summarized in the following.

Lemma 4. *Let λ_i, ϕ_i and χ_i be the i-th eigenvalue, left eigenvector and right eigenvector of A with*

$$A\phi_i = \lambda_i\phi_i, \quad \chi_i^*A = \lambda_i\chi_i^*, \tag{18}$$

where * denotes the complex conjugate transpose. Then,

$$\Delta\lambda_i = \mathcal{J}_{\lambda_i,A}\text{vec}(\Delta A), \quad \Delta\phi_i = \mathcal{J}_{\phi_i,A}\text{vec}(\Delta A),$$

where $\mathcal{J}_{\lambda_i,A} \in \mathbb{C}^{1\times n^2}$, $\mathcal{J}_{\phi_i,A} \in \mathbb{C}^{n\times n^2}$, with

$$\mathcal{J}_{\lambda_i,A} \stackrel{\text{def}}{=} \frac{1}{\chi_i^*\phi_i}(\phi_i^T \otimes \chi_i^*), \quad \mathcal{J}_{\phi_i,A} \stackrel{\text{def}}{=} (\lambda_i I_n - A)^\dagger\left(\phi_i^T \otimes \left(I_n - \frac{\phi_i\chi_i^*}{\chi_i^*\phi_i}\right)\right). \tag{19}$$

Lemma 5. *Let λ_i and ϕ_i be the i-th eigenvalue and left eigenvector of A and $\tilde{\lambda}_i \stackrel{\text{def}}{=} \ln(\lambda_i)/\tau = (b_i + a_i\mathrm{i})/\tau$ the eigenvalue of the corresponding continuous-time state transition matrix as in* (4)*. Let furthermore the natural frequency f_i and the damping ratio ρ_i be given in* (4)*, and suppose that the element k of the mode shape φ_i is scaled to unity, i.e. $\varphi_i = C\phi_i/(C\phi_i)_k$. Then,*

$$\Delta f_i = \mathcal{J}_{f_i,A}\mathrm{vec}(\Delta A), \quad \Delta\rho_i = \mathcal{J}_{\rho_i,A}\mathrm{vec}(\Delta A), \quad \Delta\varphi_i = \mathcal{J}_{\varphi_i,A,C}\begin{bmatrix}\mathrm{vec}(\Delta A)\\ \mathrm{vec}(\Delta C)\end{bmatrix},$$

where $\mathcal{J}_{f_i,A}, \mathcal{J}_{\rho_i,A} \in \mathbb{R}^{1\times n^2}$, $\mathcal{J}_{\varphi_i,A,C} \in \mathbb{C}^{r\times(n^2+rn)}$, with

$$\begin{bmatrix}\mathcal{J}_{f_i,A}\\ \mathcal{J}_{\rho_i,A}\end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix}\mathcal{J}_{f_i,\lambda_i}\\ \mathcal{J}_{\rho_i,\lambda_i}\end{bmatrix}\begin{bmatrix}\Re(\mathcal{J}_{\lambda_i,A})\\ \Im(\mathcal{J}_{\lambda_i,A})\end{bmatrix}, \quad \mathcal{J}_{\varphi_i,A,C} \stackrel{\text{def}}{=} \frac{1}{(C\phi_i)_k}\left(I_r - \begin{bmatrix}0_{r,k-1} & \varphi_i & 0_{r,r-k}\end{bmatrix}\right)\begin{bmatrix}C\mathcal{J}_{\phi_i,A} & \phi_i^T \otimes I_r\end{bmatrix},$$

and

$$\begin{bmatrix}\mathcal{J}_{f_i,\lambda_i}\\ \mathcal{J}_{\rho_i,\lambda_i}\end{bmatrix} \stackrel{\text{def}}{=} \frac{1}{\tau|\lambda_i|^2\,|\tilde{\lambda}_i|}\begin{bmatrix}1/(2\pi) & 0\\ 0 & 100/|\tilde{\lambda}_i|^2\end{bmatrix}\begin{bmatrix}\Re(\tilde{\lambda}_i) & \Im(\tilde{\lambda}_i)\\ -\Im(\tilde{\lambda}_i)^2 & \Re(\tilde{\lambda}_i)\Im(\tilde{\lambda}_i)\end{bmatrix}\begin{bmatrix}\Re(\lambda_i) & \Im(\lambda_i)\\ -\Im(\lambda_i) & \Re(\lambda_i)\end{bmatrix} \in \mathbb{R}^{2\times 2},$$

where the real and the imaginary part of a variable are denoted by $\Re(\cdot)$ and $\Im(\cdot)$, respectively.

Finally, the covariances of the modal parameters can be computed from

$$\begin{aligned}\mathrm{cov}\left(\begin{bmatrix}f_i\\ \rho_i\end{bmatrix}\right) &= \begin{bmatrix}\mathcal{J}_{f_i,A} & 0_{1,rn}\\ \mathcal{J}_{\rho_i,A} & 0_{1,rn}\end{bmatrix}\Sigma_{A,C}\begin{bmatrix}\mathcal{J}_{f_i,A} & 0_{1,rn}\\ \mathcal{J}_{\rho_i,A} & 0_{1,rn}\end{bmatrix}^T,\\ \mathrm{cov}\left(\begin{bmatrix}\Re(\varphi_i)\\ \Im(\varphi_i)\end{bmatrix}\right) &= \begin{bmatrix}\Re(\mathcal{J}_{\varphi_i,A,C})\\ \Im(\mathcal{J}_{\varphi_i,A,C})\end{bmatrix}\Sigma_{A,C}\begin{bmatrix}\Re(\mathcal{J}_{\varphi_i,A,C})\\ \Im(\mathcal{J}_{\varphi_i,A,C})\end{bmatrix}^T,\end{aligned} \tag{20}$$

where $\Sigma_{A,C}$ is defined in (17).

5 A fast algorithm for the computation of covariance estimates

In this section, a fast algorithm for the covariance computation of the modal parameters is derived by a mathematical reformulation of the algorithm from the previous section.

5.1 Factorization of $\Sigma_{\mathcal{H}}$

First, consider a decomposition $\Sigma_{\mathcal{H}} = TT^T$ of the covariance of the subspace matrix, where the matrix T is a matrix square root of $\Sigma_{\mathcal{H}}$. This decomposition will be useful in the following sections for an efficient computation of the covariances of the system matrices or modal parameters. Computing $\Sigma_{\mathcal{H}}$ in Section 3.2, the matrix T can be obtained as follows. It holds

$$\Sigma_{\mathcal{H}} = \frac{1}{n_b(n_b-1)}\sum_{j=1}^{n_b}(h_j - \bar{h})(h_j - \bar{h})^T,$$

using samples $h_j = \mathrm{vec}(\widehat{\mathcal{H}}_j)$, $j = 1,\ldots,n_b$, where $\bar{h} \stackrel{\text{def}}{=} \frac{1}{n_b}\sum_{j=1}^{n_b} h_j$ (cf. (13)). Then, defining

$$T \stackrel{\text{def}}{=} \frac{1}{\sqrt{n_b(n_b-1)}}\begin{bmatrix}\tilde{h}_1 & \tilde{h}_2 & \ldots & \tilde{h}_{n_b}\end{bmatrix} \text{ with } \tilde{h}_k \stackrel{\text{def}}{=} h_k - \bar{h},$$

it follows $\Sigma_{\mathcal{H}} = TT^T$, where $T \in \mathbb{R}^{(p+1)rqr_0\times n_b}$. The matrix T is directly obtained from the samples of the subspace matrix on the data blocks without any additional computational cost and in the following the computation of $\Sigma_{\mathcal{H}}$ itself is avoided. Moreover, the number of blocks n_b is often limited in practice due to available data length, with usually $n_b < (p+1)rqr_0$, which means that matrix T has much less columns than $\Sigma_{\mathcal{H}}$.

5.2 Sensitivity derivation of the system matrices A and C

The sensitivity computation on $\mathcal{O}$ is based on Lemma 2, where the sensitivities of the ith left and right singular vectors are obtained simultaneously as $B_i^\dagger C_i$. In the following, an alternative computation is proposed to compute these sensitivities separately, as only the *left* singular vectors are needed for $\mathcal{O}$. The new computation makes use of the block structure of B_i [9]. It avoids the costly computation of the pseudoinverse of B_i and uses the inversion of a smaller matrix instead.

Proposition 6. *Define*

$$K_i \stackrel{\text{def}}{=} \left(I_{qr_0} + \begin{bmatrix} 0_{qr_0-1,qr_0} \\ 2v_i^T \end{bmatrix} - \frac{\mathcal{H}^T\mathcal{H}}{\sigma_i^2} \right)^{-1}, \tag{21}$$

$$\widetilde{B}_{i,1} \stackrel{\text{def}}{=} \begin{bmatrix} I_{(p+1)r} + \frac{\mathcal{H}}{\sigma_i} K_i \left(\frac{\mathcal{H}^T}{\sigma_i} - \begin{bmatrix} 0_{qr_0-1,(p+1)r} \\ u_i^T \end{bmatrix} \right) & \frac{\mathcal{H}}{\sigma_i} K_i \end{bmatrix}, \tag{22}$$

$$\widetilde{B}_{i,2} \stackrel{\text{def}}{=} \begin{bmatrix} K_i \left(\frac{\mathcal{H}^T}{\sigma_i} - \begin{bmatrix} 0_{qr_0-1,(p+1)r} \\ u_i^T \end{bmatrix} \right) & K_i \end{bmatrix}, \tag{23}$$

$$\widetilde{C}_i \stackrel{\text{def}}{=} \frac{1}{\sigma_i} \begin{bmatrix} (I_{(p+1)r} - u_i u_i^T)(v_i^T \otimes I_{(p+1)r}) \\ (I_{qr_0} - v_i v_i^T)(I_{qr_0} \otimes u_i^T) \end{bmatrix}. \tag{24}$$

Then, a perturbation on $\mathcal{H}$ is propagated to the left and right singular vectors by

$$\operatorname{vec}(\Delta U_1) = \begin{bmatrix} \widetilde{B}_{1,1}\widetilde{C}_1 \\ \vdots \\ \widetilde{B}_{n,1}\widetilde{C}_n \end{bmatrix} \operatorname{vec}(\Delta\mathcal{H}), \quad \operatorname{vec}(\Delta V_1) = \begin{bmatrix} \widetilde{B}_{1,2}\widetilde{C}_1 \\ \vdots \\ \widetilde{B}_{n,2}\widetilde{C}_n \end{bmatrix} \operatorname{vec}(\Delta\mathcal{H}). \tag{25}$$

With the sensitivities of the singular vectors, the sensitivity of the observability matrix is obtained efficiently in the following corollary.

Corollary 7. *Let $\widetilde{B}_{i,1}$ and $\widetilde{C}_i$ be given in Lemma 6 for $i = 1, \ldots, n$. Then, a first-order perturbation on $\mathcal{H}$ is propagated to the observability matrix by*

$$\operatorname{vec}(\Delta\mathcal{O}) = \widetilde{\mathcal{J}}_{\mathcal{O},\mathcal{H}} \operatorname{vec}(\Delta\mathcal{H}),$$

where

$$\widetilde{\mathcal{J}}_{\mathcal{O},\mathcal{H}} \stackrel{\text{def}}{=} \frac{1}{2} \begin{bmatrix} \sigma_1^{-1/2} u_1 (v_1 \otimes u_1)^T \\ \vdots \\ \sigma_n^{-1/2} u_n (v_n \otimes u_n)^T \end{bmatrix} + \begin{bmatrix} \sigma_1^{1/2}\, \widetilde{B}_{1,1}\widetilde{C}_1 \\ \vdots \\ \sigma_n^{1/2}\, \widetilde{B}_{n,1}\widetilde{C}_n \end{bmatrix}.$$

In [10] further details are given for an efficient multiplication of $\widetilde{\mathcal{J}}_{\mathcal{O},\mathcal{H}} T$.

5.3 Covariance computation of modal parameters

Now, all the bricks are available for computing the covariance of the modal parameters. When estimating the covariance of the subspace matrix in Section 5.1, only the matrix T is computed that satisfies the decomposition $\Sigma_{\mathcal{H}} = TT^T$. From (17) and (20) follows then

$$\operatorname{cov}\left(\begin{bmatrix} f_i \\ \rho_i \end{bmatrix} \right) = U_{f_i,\rho_i} U_{f_i,\rho_i}^T, \quad \operatorname{cov}\left(\begin{bmatrix} \Re(\varphi_i) \\ \Im(\varphi_i) \end{bmatrix} \right) = U_{\varphi_i} U_{\varphi_i}^T, \tag{26}$$

where the quantities

$$U_{f_i,\rho_i} \stackrel{\text{def}}{=} \begin{bmatrix} \mathcal{J}_{f_i,A} \\ \mathcal{J}_{\rho_i,A} \end{bmatrix} \mathcal{J}_{A,\mathcal{O}} \widetilde{\mathcal{J}}_{\mathcal{O},\mathcal{H}} T, \quad U_{\varphi_i} \stackrel{\text{def}}{=} \begin{bmatrix} \Re(\mathcal{J}_{\varphi_i,A,C}) \\ \Im(\mathcal{J}_{\varphi_i,A,C}) \end{bmatrix} \begin{bmatrix} \mathcal{J}_{A,\mathcal{O}} \\ \mathcal{J}_{C,\mathcal{O}} \end{bmatrix} \widetilde{\mathcal{J}}_{\mathcal{O},\mathcal{H}} T, \tag{27}$$

need to be computed first. Note also that in the products $\mathcal{J}_{\lambda_i,A}\mathcal{J}_{A,\mathcal{O}}$ and $\mathcal{J}_{\phi_i,A}\mathcal{J}_{A,\mathcal{O}}$, which appear in the computation of U_{f_i,ρ_i} and U_{φ_i}, one can combine Kronecker products avoiding the computation of big matrices. This is further detailed in [10].

6 Applications

The computation times between an implementation of the original algorithm from [19] and the fast algorithm in Section 5 are compared on two test cases. The algorithms are tested on an Intel Xeon CPU 3.40 GHz with 16 GByte in Matlab 7.10.0.499. The parameters are set as follows:

- The covariance-driven subspace matrix $\mathcal{H}$ of size $(p+1)r \times qr_0$ is built from the data, where $p+1 = q$ is chosen, as recommended in [1].
- The number of reference sensors is $r_0 = 3$ in both test cases.
- The model order is set to $n = qr_0$.
- The number of data blocks used for the covariance computation is $n_b = 200$.

To compare the performance of the algorithms, the modal analysis and covariance computation is done for different model orders n by choosing $q = 2, \ldots, 70$ for the subspace matrix.

6.1 Z24 Bridge

The proposed algorithms have been applied on vibrational data of the Z24 Bridge [16], a benchmark of the COST F3 European network. The analyzed data is the response of the bridge to ambient excitation (traffic under the bridge) measured in 154 points, mainly in the vertical and at some points also the transverse and lateral directions, and sampled at 100 Hz. Altogether, nine data sets have been recorded, each covering a part of the whole structure. In this study, only the first data set is used to demonstrate the computations. It contains data from $r = 33$ sensors, from which $r_0 = 3$ reference sensors are chosen, and each signal contains 65,535 samples. The resulting computation times from both algorithms are compared in Figure 1.

It can be seen that the new fast algorithm outperforms its original version. For example, at model order $n = 72$ the computation time with the fast algorithm is 11.9 s, while the original version took 597 s. Being significantly faster, also higher model orders are feasible for memory with the new algorithm. In this case the computation was possible at model orders over 200 at computation times of less than a few minutes.

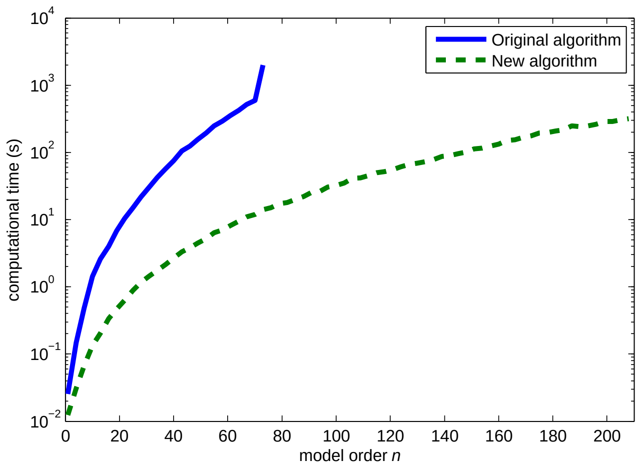

Figure 1: Computation times for identification and covariance computation of modal parameters of Z24 Bridge for different model orders n (log scale).

Table 1: Overview of the estimated modes in first measurement setup of Z24 Bridge with natural frequencies f, their variation coefficient $\tilde{\sigma}_f = \sigma_f/f \cdot 100$, the damping ratios ρ and their variation coefficient $\tilde{\sigma}_\rho = \sigma_\rho/\rho \cdot 100$.

mode	f (in Hz)	$\tilde{\sigma}_f$	ρ (in %)	$\tilde{\sigma}_\rho$
1	3.876	0.14	0.54	26
2	4.863	0.22	1.60	13
3	9.789	0.44	1.25	45
4	10.34	0.46	1.30	39
5	12.38	0.33	3.14	16
6	13.25	0.65	3.03	23
7	17.54	1.49	2.19	51
8	19.25	1.45	2.61	44
9	19.86	1.31	2.27	47
10	26.04	1.78	2.13	81

In Table 1 the identified modal parameters with their coefficients of variation (standard deviation of the parameter divided by the parameter) are shown, identified with the parameter $q = 50$. Note that only one of the nine measurement setups of Z24 Bridge is used for this identification example. Results can be improved by using all setups for the system identification and uncertainty quantification, e.g. as in [8]. The higher modes 7–10 are apparently more difficult to identify and the variation coefficients of the identified frequencies and damping ratios are subsequently higher. As expected by statistical theory [13], the uncertainties on the identified damping ratios are much higher than on the frequencies.

6.2 S101 Bridge

Within the IRIS project an extensive measurement campaign of the prestressed concrete road bridge S101 was taken, which was planned and organized by the Austrian company VCE and accomplished by VCE and the University of Tokyo [21]. For vibration measurements a BRIMOS measurement system was used, consisting in 15 sensor locations on the bridge deck, where 14 sensors were placed on ones side of the span and one sensor on the other side of the span. In each location the bridge deck's vertical, longitudinal and transversal directions were measured. Altogether, 45 acceleration sensors were applied. All values were recorded permanently with a sampling frequency of 500 Hz. During the three days measurement campaign 714 data files with 165 000 data points were produced. In this study only the first dataset is used.

Restricting the system identification to the first five modes in the frequency range [0–18 Hz], the data was downsampled from sampling rate 500 Hz by factor 5. All $r = 45$ sensors were used and $r_0 = 3$ reference sensors were chosen. The computation times for system identification and the confidence interval computation using the fast algorithm described in Section 5 are shown in Figure 2 for different model orders $n = qr_0$

Table 2: Overview of the estimated modes of S101 Bridge with natural frequencies f, their variation coefficient $\tilde{\sigma}_f = \sigma_f/f \cdot 100$, the damping ratios ρ and their variation coefficient $\tilde{\sigma}_\rho = \sigma_\rho/\rho \cdot 100$.

mode	f (in Hz)	$\tilde{\sigma}_f$	ρ (in %)	$\tilde{\sigma}_\rho$
1	4.036	0.12	0.78	15
2	6.281	0.08	0.56	20
3	9.677	0.18	1.3	14
4	13.27	0.13	1.5	13
5	15.72	0.37	1.3	17

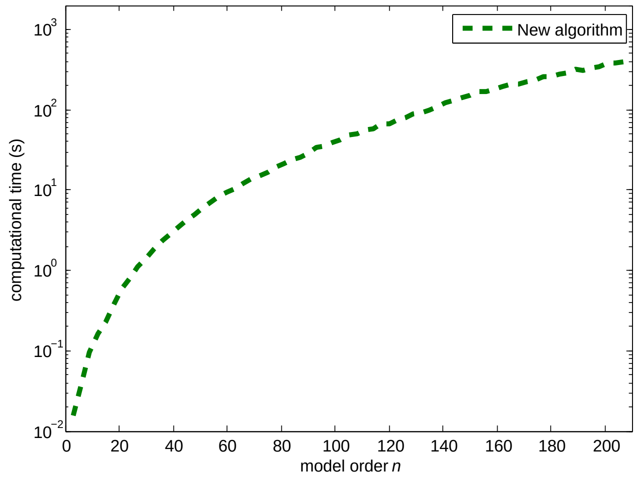

Figure 2: Computation times for identification and covariance computation of modal parameters of S101 Bridge for different model orders n (log scale).

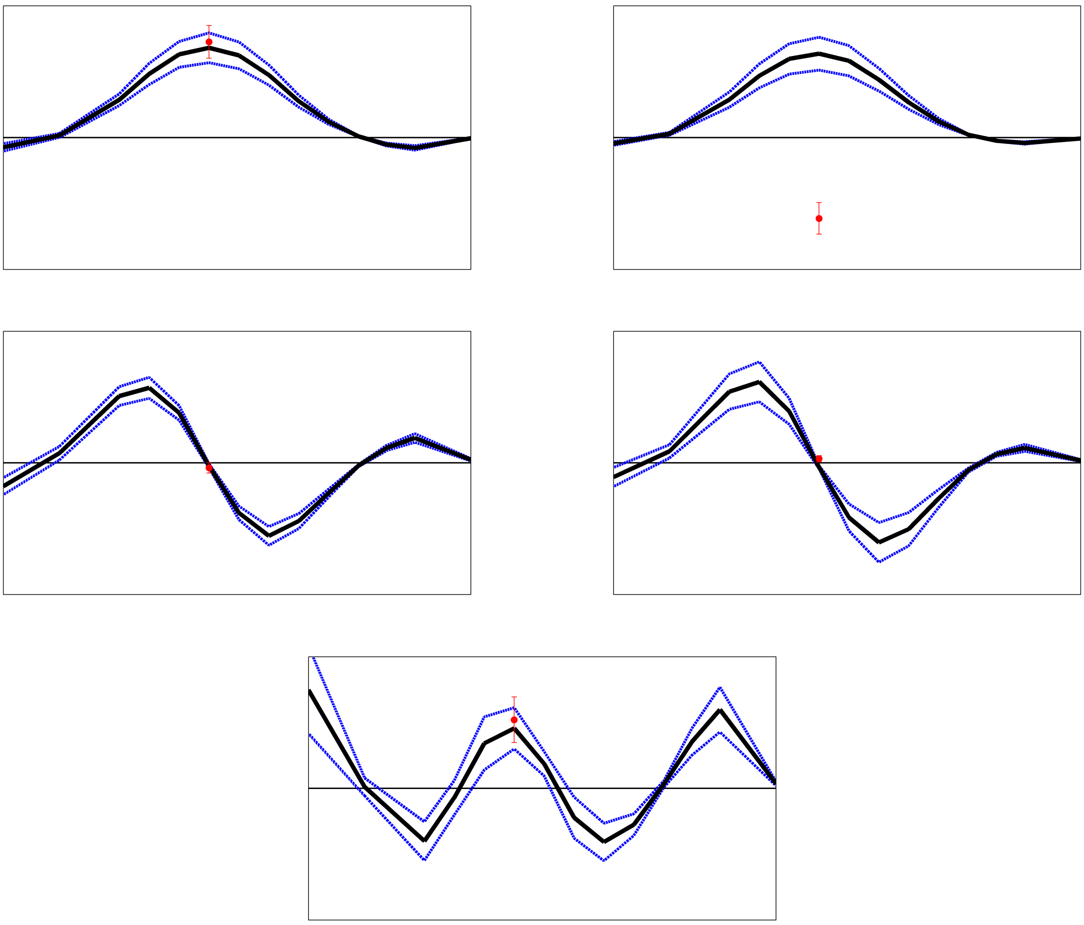

Figure 3: First five mode shapes in vertical direction on one side of the span (line) and the sensor on the other side of the span (red point) with their $\pm 2\sigma$ uncertainty bounds.

for $q = 1, \ldots, 70$.

The identified modal parameters with their variation coefficients are summarized in Table 2 and the mode shapes with their confidence bounds are presented in Figure 3. These results were obtained using parameter $q = 35$. A new scheme for the mode shape normalization has been introduced in [7] that is used in these examples.

7 Conclusions

In this paper, an algorithm was presented for the efficient computation of uncertainty bounds for system matrices A and C and associated modal parameters in stochastic subspace-based system identification (SSI), based on the algorithm described in [19]. Compared to its original derivation, a significant decrease in computation time and the feasibility of larger model orders was possible, showing the efficiency of the new computation scheme presented in this paper. Extensions to the uncertainty computation at multiple model orders for stabilization diagrams [10] and subspace identification from multiple measurement setups [8, 12] are possible.

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