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VARIANCE COMPUTATION OF MODAL PARAMETER ESTIMATES FROM UPC SUBSPACE IDENTIFICATION

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ABSTRACT

The vibration response of a structure from ambient excitation is measured and used to estimate the modal parameters in Operational Modal Analysis (OMA). Subspace-based system identification allows the accurate estimation of the modal parameters (natural frequencies, damping ratios, mode shapes) from output-only measurements, amongst others with data-driven methods like the Unweighted Principal Component (UPC) algorithm. Due to unknown excitation, measurement noise and finite measurements, all modal parameter estimates are inherently afflicted by uncertainty. The information on their uncertainty is most relevant to assess the quality of the modal parameter estimates, or when comparing modal parameters from different datasets. Previously, a method for variance estimation has been developed for the covariance-driven subspace identification. In this paper, we present an extension of this method for the variance computation of modal parameters for the UPC subspace algorithm. Developing the sensitivities of the modal parameters with respect to the output covariances, the uncertainty is propagated from the measurements to the modal parameters from UPC in a rigorous way. The resulting variance expressions are easy to evaluate and computationally tractable when using an efficient implementation. In a second step, the uncertainty information of the stabilization diagram is used to extract appropriately weighted global mode estimates and their variance. The method is applied to experimental data from the Z24 Bridge.

Keywords: subspace methods, uncertainty estimation, variance analysis

1. INTRODUCTION

The goal of operational modal analysis (OMA) is the identification of modal parameters (frequencies, damping ratios, mode shapes) from output-only vibration measurements on a structure under unknown ambient excitation (e.g. wind, traffic). The modal parameters are related to the eigenvalues and observed eigenvectors of a linear time-invariant system. Many methods for OMA are available in the literature [1]. Subspace-based linear system identification algorithms are state of the art, fitting a linear model to output-only measurements taken from a system [2, 3]. In this paper we focus on the data-driven Unweighted Principal Component (UPC) subspace method. It is based on the orthogonal projection of data Hankel matrices containing lags of the measurement data. The purpose of this paper is the variance analysis of modal parameters obtained from this method.

Using noisy measurement data, subspace algorithms provide parameter estimates that are afflicted with statistical uncertainty due to finite data, unknown inputs and sensor noise properties. In the field of vibration analysis, explicit expressions for their variance estimation have been proposed for some subspace methods. A successful strategy has been to propagate an estimated covariance on the measurements to the desired parameters based on a sensitivity analysis. The required sensitivities are derived analytically through the propagation of a first-order perturbation from the data to the identified parameters. This approach has the advantage of computational convenience: the sample covariance as the starting point is directly linked to the measurements and therefore easy to compute, and the sensitivities are computed using the system identification estimates. In [4], details of this scheme are developed for the covariance computation for output-only covariance-driven subspace identification.

Besides technical correctness, this approach yields a well-adapted fast and memory efficient implementation in the covariance-driven output-only case [5]. As a result, the computational cost for the variance computation of an entire stabilization diagram analysis is reduced significantly. This was mandatory for realistic applications on large structures like in [6, 7]. Very recently, the generalization of this approach to a wider class of subspace algorithms in both output-only and input/output frameworks, and for covariance-driven and data-driven methods, has been presented in [8].

A challenge in the variance estimation for data-driven methods is the nature of the data Hankel matrices that does not allow an intrinsic covariance formulation as this is the case with covariance-driven subspace methods. Instead, the variance estimation for data-driven algorithms is carried out by developing an equivalent formulation in a covariance-driven form, where the sensitivities of the modal parameters with respect to output covariances are derived, and the uncertainty of the output covariances is propagated to the modal parameters in a rigorous way.

In a second step, the uncertainty information of the stabilization diagram is used to extract appropriately weighted global mode estimates and their variance.

2. THE SUBSPACE METHOD

2.1. Mechanical and state space models

The vibration behavior of a linear time-invariant mechanical structure, which is observed at some sensor positions, can be described by the equations

$$\begin{cases} \mathcal{M}\ddot{q}(t) + \mathcal{C}\dot{q}(t) + \mathcal{K}q(t) = \tilde{u}(t) \\ \qquad\qquad\qquad\qquad y(t) = L_a\ddot{q}(t) + L_v\dot{q}(t) + L_d q(t) + v(t) \end{cases} \tag{1}$$

where $\mathcal{M}$, $\mathcal{C}$ and $\mathcal{K} \in \mathbb{R}^{m\times m}$ are mass, stiffness and damping matrices, and m is the number of degrees of freedom. The vector $q(t) \in \mathbb{R}^m$ contains the displacements at the degrees of freedom generated by the unknown inputs $\tilde{u}(t) \in \mathbb{R}^m$. The vector $y(t) \in \mathbb{R}^r$ contains the observed outputs, with r being the number of sensors. The matrices L_a, L_v and $L_d \in \mathbb{R}^{N_o\times m}$ represent how accelerations, velocities and displacements are obtained from the model degrees of freedom. The vector $v(t) \in \mathbb{R}^r$ is the sensor noise. Both $\tilde{u}(t)$ and $v(t)$ are assumed to be white noise with finite fourth moments and uncorrelated with the known inputs.

Sampling Eq. (1) at rate τ yield the discrete-time state space representation

$$\begin{cases} x_{k+1} = Ax_k + w_k \\ \quad\; y_k = Cx_k + v_k, \end{cases} \tag{2}$$

where $x_k = [\dot{q}(k\tau)^T \;\; q(k\tau)^T]^T \in \mathbb{R}^n$ is the state vector, $n = 2m$ is the model order and

$$A_c = \begin{bmatrix} -\mathcal{M}^{-1}\mathcal{C} & -\mathcal{M}^{-1}\mathcal{K} \\ I_m & 0_{m,m} \end{bmatrix} \in \mathbb{R}^{n\times n},\; A = \exp(A_c\tau), \tag{3}$$

$$C = \begin{bmatrix} L_v - L_a\mathcal{M}^{-1}\mathcal{C} & L_d - L_a\mathcal{M}^{-1}\mathcal{K} \end{bmatrix} \in \mathbb{R}^{r\times n}, \tag{4}$$

are the state transition and output matrices, respectively. The state noise term w_k is linked to the unknown inputs $\tilde{u}(t)$, and the output noise term v_k has contributions from $v(t)$ and in the case of acceleration measurements also from $\tilde{u}(t)$.

The modal parameters of system (1) are equivalently found in system (2) as follows. Let λ_i and ϕ_i be eigenvalues and eigenvectors of A, for $i = 1, \ldots, n$. Then the eigenvalues μ_i of system (1), modal frequencies f_i, damping ratios ξ_i and mode shapes φ_i are obtained by

$$\mu_i = \frac{\log(\lambda_i)}{\tau},\;\; f_i = \frac{|\mu_i|}{2\pi},\;\; \xi_i = \frac{-\mathrm{Re}(\mu_i)}{|\mu_i|},\;\; \varphi_i = C\phi_i. \tag{5}$$

2.2. Subspace identification

The UPC subspace algorithm is based on projections of output data Hankel matrices with a "past" and "future" time horizon. These projections are designed in a way that the column space of the resulting matrix $\mathcal{H}$ is defined by the observability matrix $\Gamma_p = [C^T \;\; \ldots \;\; (CA^{p-1})^T]^T$ of system (2).

The data Hankel matrix $\mathcal{Y}_{i|j}$ is defined for the samples $y_l \in \mathbb{R}^r$ as

$$\mathcal{Y}_{i|j} \stackrel{\text{def}}{=} \begin{bmatrix} y_i & y_{i+1} & \cdots & y_{i+N-1} \\ \vdots & \vdots & \ddots & \vdots \\ y_j & y_{j+1} & \cdots & y_{j+N-1} \end{bmatrix} \in \mathbb{R}^{(j-i+1)r\times N}. \tag{6}$$

Let $N + p + q$ be the number of available samples for the outputs y_k, where q and p are parameters that define a "past" and "future" time horizon. They are most often equal, and assumed to be large enough to satisfy the condition $\min\{(p-1)r, qr\} \geq n$. The "past" ($^-$) and "future" ($^+$) data Hankel matrices containing the outputs are defined as

$$\mathcal{Y}^- \stackrel{\text{def}}{=} \frac{1}{\sqrt{N}}\mathcal{Y}_{0|q-1},\; \mathcal{Y}^+ \stackrel{\text{def}}{=} \frac{1}{\sqrt{N}}\mathcal{Y}_{q|q+p-1}. \tag{7}$$

The UPC method is defined by the orthogonal projection

$$\mathcal{H} = \mathcal{Y}^+/\mathcal{Y}^- = \mathcal{Y}^+\mathcal{Y}^{-^T}\left(\mathcal{Y}^-\mathcal{Y}^{-^T}\right)^\dagger \mathcal{Y}^-,$$

fulfilling the factorization property $\mathcal{H} = \Gamma_p\mathcal{Z}$ into observability matrix Γ_p and matrix $\mathcal{Z}$ which is a matrix containing the Kalman filter states. Note that in practice the observability matrix Γ_p is obtained through the LQ decomposition

$$\begin{bmatrix} \mathcal{Y}^- \\ \mathcal{Y}^+ \end{bmatrix} = \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$$

from a singular value decomposition (SVD) of L_{21}, since it holds $\mathcal{H} = L_{21}Q_1$ where Q_1 is an orthonormal matrix.

From the SVD

$$L_{21} = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}$$

and its truncation at the desired model order n, the observability matrix is estimated from $\Gamma_p = U_1 S_1^{1/2}$. The output matrix C is then obtained by direct extraction of the first block row of Γ_p. The state transition matrix A is obtained from the shift invariance property of Γ_p as the least squares solution $A = \Gamma_\uparrow^\dagger \Gamma_\downarrow$, where $\Gamma_\uparrow$ and $\Gamma_\downarrow$ are obtained from Γ_p by removing the last and first block row, respectively. Finally, the modal parameters are obtained from (5).

3. VARIANCE ESTIMATION

In [4, 5], a method for the variance estimation of modal parameters was developed for the output-only covariance-driven subspace algorithm. In [8], this method was extended for the data-driven UPC algorithm, amongst other data-driven and input/output algorithms, which is presented in the following.

The computation of the modal parameter covariance results from the propagation of sample covariances on auto-covariance estimates of the outputs through all steps of the modal identification algorithm. These sample covariances reflect in particular the unknown inputs due to non-measurable excitation sources and the sensor noise, and they contribute in a non-trivial way to the covariance of the modal parameter estimates. The propagation to the modal parameter estimates is based on the delta method [9], where the analytical sensitivity matrices are obtained from perturbation theory [4].

Let ΔX be a first-order perturbation of a matrix-valued variable X. Then, for a function $Y = f(X)$ it holds $\text{vec}(\Delta Y) = \mathcal{J}_{Y,X}\,\text{vec}(\Delta X)$, where $\mathcal{J}_{Y,X} = \partial\text{vec}(f(X))/\partial\text{vec}(X)$, where $\text{vec}(\cdot)$ denotes the column stacking vectorization operator. Subsequently, covariance expressions for the estimates satisfy

$$\text{cov}(\text{vec}(\widehat{Y})) \approx \widehat{\mathcal{J}}_{Y,X}\text{cov}(\text{vec}(\widehat{X}))\widehat{\mathcal{J}}_{Y,X}^{T}. \tag{8}$$

For simplicity of notation we dismiss the notation $\widehat{\ }$ for an estimate in the following.

3.1. From data-driven to an equivalent covariance-driven formulation

In contrast to the covariance-driven algorithms, the number of columns of matrix $\mathcal{H}$ depends on the number of data samples N for the data-driven algorithms. Hence, the matrix $\mathcal{H}$ does not converge to a fixed limit for data-driven algorithms for $N \to \infty$, which is a problem for the subsequent covariance analysis. Instead, an equivalent covariance-driven form can be defined by

$$\mathcal{H}_{\text{cov}} = \mathcal{H}\mathcal{H}^T.$$

The resulting covariance-driven algorithm defined by $\mathcal{H}_{\text{cov}}$ yields *identical* estimates of the observability matrix Γ_p as the original data-driven algorithm. It is easy to see that both matrices $\mathcal{H}_{\text{cov}}$ and $\mathcal{H}$ indeed have the same column space: let the thin SVD of $\mathcal{H} = USV^T$ be given, then $\mathcal{H}_{\text{cov}} = \mathcal{H}\mathcal{H}^T = US^2U^T$, and hence both $\mathcal{H}_{\text{cov}}$ and $\mathcal{H}$ have the same left singular vectors (up to a change of basis). Hence, the variance of the resulting modal parameter estimates is also identical between both algorithms.

From the definition of the UPC algorithm it follows thus

$$\mathcal{H}_{\text{cov}} = \left(\mathcal{Y}^+\mathcal{Y}^{-^T}\right)\left(\mathcal{Y}^-\mathcal{Y}^{-^T}\right)^{\dagger}\left(\mathcal{Y}^+\mathcal{Y}^{-^T}\right)^T. \tag{9}$$

Note that

$$\mathcal{R}_+ = \mathcal{Y}^+\mathcal{Y}^{-^T},\ \ \mathcal{R}_- = \mathcal{Y}^-\mathcal{Y}^{-^T} \tag{10}$$

are matrices containing output covariances. For these matrices it is easy to obtain a sample covariance estimate, as detailed in the following section.

3.2. Sample covariance estimation

The starting point of the variance propagation to the modal parameters is the covariance of and between the auto-covariance matrices that are involved in the subspace algorithm, i.e. $\mathcal{R}_+$ and $\mathcal{R}_-$. In particular we require the estimation of $\text{cov}(\text{vec}(\mathcal{R}_+))$, $\text{cov}(\text{vec}(\mathcal{R}_-))$ and $\text{cov}(\text{vec}(\mathcal{R}_+), \text{vec}(\mathcal{R}_-))$. This computation follows the lines of the output-only covariance-driven algorithm, i.e. for $\text{cov}(\text{vec}(\mathcal{R}_+))$, as described in detail in [4, 5], and is generalized in [8] as follows.

Divide the data matrices $\mathcal{Y}^+$ and $\mathcal{Y}^-$ into n_b blocks and normalize them with respect to their length, such that

$$\sqrt{N}\,\mathcal{Y}^+ = \sqrt{N_b}\,\begin{bmatrix}\mathcal{Y}_1^+ & \mathcal{Y}_2^+ & \dots & \mathcal{Y}_{n_b}^+\end{bmatrix},\ \ \sqrt{N}\,\mathcal{Y}^- = \sqrt{N_b}\,\begin{bmatrix}\mathcal{Y}_1^- & \mathcal{Y}_2^- & \dots & \mathcal{Y}_{n_b}^-\end{bmatrix}, \tag{11}$$

where each block $\mathcal{Y}_k^+$ and $\mathcal{Y}_k^-$ may have the same length N_b, with $n_b \cdot N_b = N$ for simplicity. Each block may be long enough to assume statistical independence between the blocks. On each of these blocks, the respective auto-covariance estimate can be computed as $\mathcal{R}_+^k = \mathcal{Y}_k^+ \mathcal{Y}_k^{-T}$ and $\mathcal{R}_-^k = \mathcal{Y}_k^- \mathcal{Y}_k^{-T}$, which can be assumed to be i.i.d., yielding

$$\mathcal{R}_+ = \frac{1}{n_b} \sum_{k=1}^{n_b} \mathcal{R}_+^k, \quad \mathcal{R}_- = \frac{1}{n_b} \sum_{k=1}^{n_b} \mathcal{R}_-^k \,. \tag{12}$$

It follows $\text{cov}\left(\text{vec}(\mathcal{R}_*)\right) = \frac{1}{n_b} \text{cov}\left(\text{vec}(\mathcal{R}_*^k)\right)$, and the covariance between the auto-covariance matrices can be computed from the usual sample covariance as

$$\text{cov}(\text{vec}(\mathcal{R}_i), \text{vec}(\mathcal{R}_j)) = \frac{1}{n_b(n_b-1)} \sum_{k=1}^{n_b} \left(\text{vec}(\mathcal{R}_i^k) - \text{vec}(\mathcal{R}_i)\right) \left(\text{vec}(\mathcal{R}_j^k) - \text{vec}(\mathcal{R}_j)\right)^T \tag{13}$$

where $i, j \in \{+, -\}$.

3.3. Sensitivity and covariance of $\mathcal{H}$

In this section, the sensitivity of $\mathcal{H}$ is developed with respect to the underlying auto-covariance matrices in their computation. It holds, from Eq. (9), $\mathcal{H} = \mathcal{R}_+ \mathcal{R}_-^\dagger \mathcal{R}_+^T$, and thus [8]

$$\begin{aligned} \Delta \mathcal{H}_{\text{cov}} &= \Delta \mathcal{R}_+ \mathcal{R}_-^\dagger \mathcal{R}_+^T - \mathcal{R}_+ \mathcal{R}_-^\dagger \Delta \mathcal{R}_- \mathcal{R}_-^\dagger \mathcal{R}_+^T + \mathcal{R}_+ \mathcal{R}_-^\dagger \Delta \mathcal{R}_+^T, \\ \text{vec}\left(\Delta \mathcal{H}_{\text{cov}}\right) &= \mathcal{J}_{\mathcal{H},\mathcal{R}} \begin{bmatrix} \text{vec}\left(\Delta \mathcal{R}_+\right) \\ \text{vec}\left(\Delta \mathcal{R}_-\right) \end{bmatrix}, \end{aligned}$$

where $\mathcal{J}_{\mathcal{H},\mathcal{R}} = \begin{bmatrix} \mathcal{R}_+ \mathcal{R}_-^\dagger \otimes I_{pr} + (I_{pr} \otimes \mathcal{R}_+ \mathcal{R}_-^\dagger) \mathcal{P}_{pr,qr} & -\mathcal{R}_+ \mathcal{R}_-^\dagger \otimes \mathcal{R}_+ \mathcal{R}_-^\dagger \end{bmatrix}$ and $\mathcal{P}_{pr,qr}$ is a permutation matrix [8]. Then, it follows for the covariance

$$\text{cov}(\text{vec}(\mathcal{H})) = \mathcal{J}_{\mathcal{H},\mathcal{R}} \, \text{cov}\left(\begin{bmatrix} \text{vec}(\mathcal{R}_+) \\ \text{vec}(\mathcal{R}_-) \end{bmatrix} \right) \mathcal{J}_{\mathcal{H},\mathcal{R}}^T, \tag{14}$$

where the blocks for the covariance matrix on the right side are computed in Eq. (13).

3.4. Covariance of modal parameters

From $\mathcal{H}$, the modal parameters are obtained as described in Section 2.2.. The respective propagation of the covariance of $\text{vec}(\mathcal{H})$ to the modal parameters has been described in detail in [4, 5].

4. UNCERTAINTY EVALUATION FOR STABILIZATION DIAGRAM

In the previous section, the computational framework is given for variance estimation for the modal parameters from the data-driven UPC subspace algorithm. Together with the framework of the fast uncertainty computation algorithms in [5], a computationally efficient method has been developed for the uncertainty estimation at different model orders for the entire stabilization diagram. Now the question arises how to evaluate the global uncertainty of each mode, based on the alignments and their uncertainties over different model orders.

Assume that the mode alignments have been correctly extracted from stable parts of the stabilization diagram, and that biased estimates are rejected at model orders that are not stable. For each mode alignment, the problem is then to obtain one estimate of the respective mode (frequency and damping ratio) from the estimates at the different model orders, taking into account their uncertainties.

There are two connected steps:

1. Computation of the global mode estimate as a weighted mean of the estimates at the different model orders. Weights are the covariance of each mode estimate.
2. Computation of the covariance of the global mode estimate.

Note that all mode estimates in each alignment are strongly correlated since they are computed on the same data. So the second step needs to take these correlations between the different model orders into account.

4.1. Uncertainty of each mode estimate

Each mode i at any model order is described by its frequency f_i and damping ratio ξ_i. They are a function of the measured output data and in particular of the output data covariances $\mathcal{R}_1$ and $\mathcal{R}_2$, and a resulting projected Hankel matrix $\mathcal{H}$ as explained in the previous section. In particular, the sensitivities $\mathcal{J}_{f_i,\mathcal{R}}$ and $\mathcal{J}_{\xi_i,\mathcal{R}}$ of each frequency f_i and damping ratio ξ_i with respect to $\mathcal{R}_1$ and $\mathcal{R}_2$ yields an analytical relation based on the formulas above, leading to

$$\begin{bmatrix}\Delta f_i \\ \Delta \xi_i\end{bmatrix} = \begin{bmatrix}\mathcal{J}_{f_i,\mathcal{R}} \\ \mathcal{J}_{\xi_i,\mathcal{R}}\end{bmatrix} \begin{bmatrix}\text{vec}\left(\Delta\mathcal{R}_+\right) \\ \text{vec}\left(\Delta\mathcal{R}_-\right)\end{bmatrix}. \tag{15}$$

Then, the 2x2 covariance matrix of each mode (combined for frequency and damping ratio) is actually computed through

$$\Sigma_i = \text{cov}\left(\begin{bmatrix}f_i \\ \xi_i\end{bmatrix}\right) = \begin{bmatrix}\mathcal{J}_{f_i,\mathcal{R}} \\ \mathcal{J}_{\xi_i,\mathcal{R}}\end{bmatrix} \text{cov}\left(\begin{bmatrix}\text{vec}\left(\mathcal{R}_+\right) \\ \text{vec}\left(\mathcal{R}_-\right)\end{bmatrix}\right) \begin{bmatrix}\mathcal{J}_{f_i,\mathcal{R}} \\ \mathcal{J}_{\xi_i,\mathcal{R}}\end{bmatrix}^T.$$

4.2. Weighted mean for different model orders

Let $i = 1, \ldots, n_m$ be the estimates of the same mode at different model orders. They are weighted with their covariance matrices, summed up and re-normalized, as follows:

$$\begin{bmatrix}\bar{f} \\ \bar{\xi}\end{bmatrix} = \left(\sum_{i=1}^{n_m} \Sigma_i^{-1}\right)^{-1} \left(\sum_{i=1}^{n_m} \Sigma_i^{-1} \begin{bmatrix}f_i \\ \xi_i\end{bmatrix}\right)$$

4.3. Covariance of weighted mean

Since the estimates at the different model orders are strongly correlated, this correlation needs to be taken into account when computing the covariance of the weighted mean. This correlation is indirectly considered when relating the uncertainty of the weighted mean to the uncertainty of all the modal parameters, which is then related to the uncertainty of the output covariance uncertainties $\Delta\mathcal{R}_+$ and $\Delta\mathcal{R}_-$ as a common factor. In this computation, the sensitivities of all modal parameters at the different model orders are taken into account.

We can rewrite the weighted mean as

$$\begin{bmatrix}\bar{f} \\ \bar{\xi}\end{bmatrix} = \left(\sum_{i=1}^{n_m} \Sigma_i^{-1}\right)^{-1} \begin{bmatrix}\Sigma_1^{-1} & \Sigma_2^{-1} & \ldots & \Sigma_{n_m}^{-1}\end{bmatrix} \begin{bmatrix}f_1 \\ \xi_1 \\ f_2 \\ \xi_2 \\ \vdots \\ f_{n_m} \\ \xi_{n_m}\end{bmatrix}.$$

With this equation, the uncertainty of the weighted mean can be linked to the uncertainty of the estimates at each model order, i.e.

$$\begin{bmatrix}\Delta\bar{f}\\ \Delta\bar{\xi}\end{bmatrix} = \left(\sum_{i=1}^{n_m}\Sigma_i^{-1}\right)^{-1}\begin{bmatrix}\Sigma_1^{-1} & \Sigma_2^{-1} & \dots & \Sigma_{n_m}^{-1}\end{bmatrix}\begin{bmatrix}\Delta f_1\\ \Delta\xi_1\\ \Delta f_2\\ \Delta\xi_2\\ \vdots\\ \Delta f_{n_m}\\ \Delta\xi_{n_m}\end{bmatrix}.$$

Plugging in Equation (15), the uncertainty of the weighted mean is linked to the common uncertainty of the Hankel matrix, taking into account all the dependencies between the different model orders,

$$\begin{bmatrix}\Delta\bar{f}\\ \Delta\bar{\xi}\end{bmatrix} = \underbrace{\left(\sum_{i=1}^{n_m}\Sigma_i^{-1}\right)^{-1}\begin{bmatrix}\Sigma_1^{-1} & \Sigma_2^{-1} & \dots & \Sigma_{n_m}^{-1}\end{bmatrix}\begin{bmatrix}\mathcal{J}_{f_1,\mathcal{R}}\\ \mathcal{J}_{\xi_1,\mathcal{R}}\\ \mathcal{J}_{f_2,\mathcal{R}}\\ \mathcal{J}_{\xi_2,\mathcal{R}}\\ \vdots\\ \mathcal{J}_{f_{n_m},\mathcal{R}}\\ \mathcal{J}_{\xi_{n_m},\mathcal{R}}\end{bmatrix}}_{=\mathcal{J}_{\text{all}}}\begin{bmatrix}\text{vec}\left(\Delta\mathcal{R}_+\right)\\ \text{vec}\left(\Delta\mathcal{R}_-\right)\end{bmatrix}.$$

With this relation, the covariance of the weighted mean follows readily as

$$\text{cov}\left(\begin{bmatrix}\bar{f}\\ \bar{\xi}\end{bmatrix}\right) = \mathcal{J}_{\text{all}}\,\text{cov}\left(\begin{bmatrix}\text{vec}\left(\mathcal{R}_+\right)\\ \text{vec}\left(\mathcal{R}_-\right)\end{bmatrix}\right)\mathcal{J}_{\text{all}}^T.$$

5. APPLICATION

The algorithm for the uncertainty computation has been validated previously on Monte Carlo simulations in [8], ensuring the correct computation of the uncertainty bounds in a numerical study.

In the following, we apply the algorithm to the Z24 Bridge [6, 10], a benchmark of the COST F3 European network. The analyzed data is the response of the bridge to ambient excitation (traffic under the bridge) measured in 154 points, mainly in the vertical and at some points also the transverse and lateral directions, and sampled at 100 Hz. Because at most 33 sensors were available (counting one sensor for each measured direction), 9 datasets have been recorded, each containing the measurements from 5 fixed and 28 moving sensors, except dataset 5 containing only 22 moving sensors. Like this, altogether 251

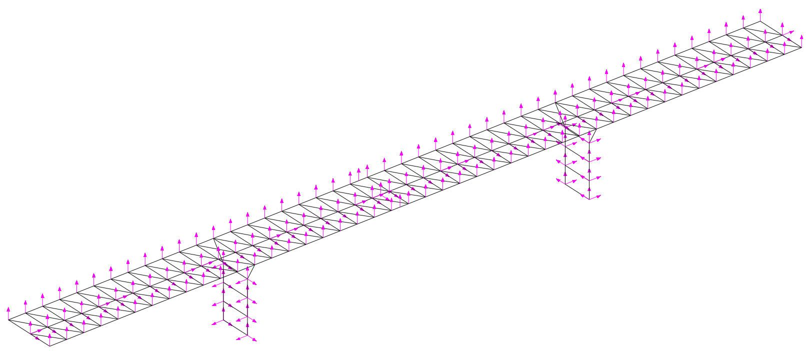

Figure 1: Model of Z24 Bridge with sensor positions and directions.

sensors were mimicked. Each signal contains 65,535 samples. In Figure 1, the model of the bridge with all sensor positions and directions is shown.

Applying the UPC algorithm on the first sensor setup of Z24 in a Matlab implementation, a stabilization diagram is obtained in Figure 2. Selecting the first frequency and the respective damping ratios, a zoom gives more detailed information on the estimates and their uncertainties over the different model orders in Figure 3. This figure also includes the weighted mean of the frequencies and damping ratios of the first mode obtained from the different model orders, as shown in Section 4..

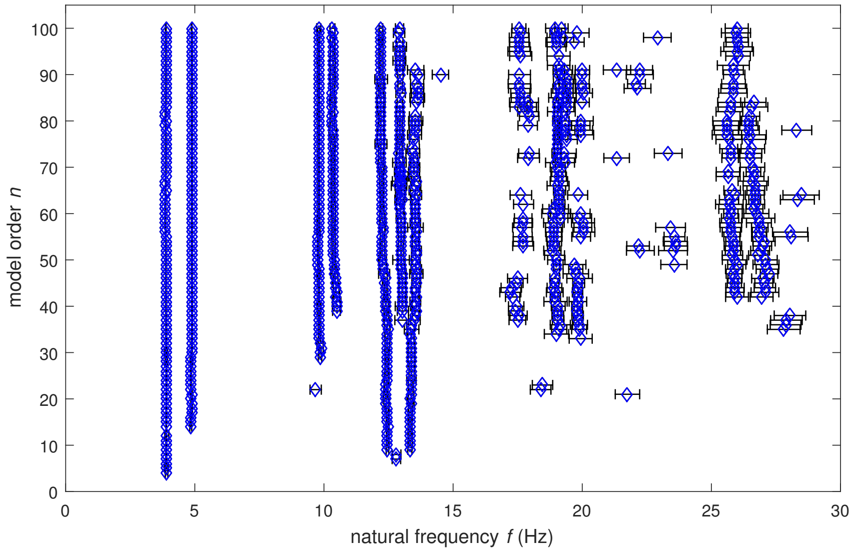

Figure 2: Stabilization diagram for first setup of Z24 Bridge with uncertainty bounds.

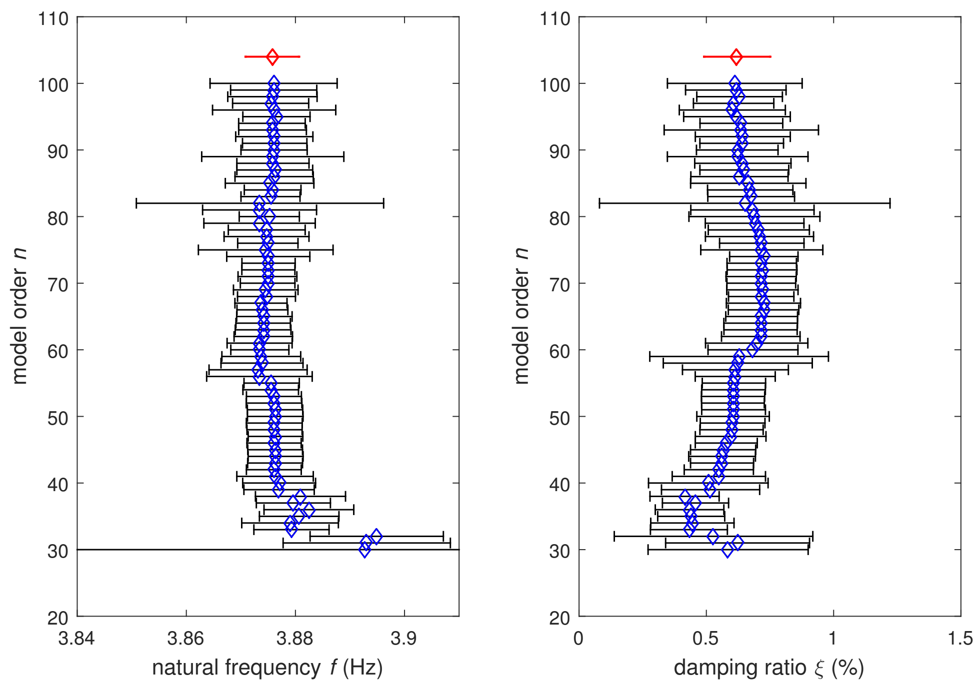

Figure 3: Zoom on frequency (left) and damping ratio (right) estimates of the first mode. The red value on top is the weighted mean over all model orders.

Finally, a stabilization diagram within an implementation in ARTeMIS Modal 5 [11] is shown in Figure 4. Confidence ellipsoids of each modal frequency and damping ratio pair can also be displayed in a

Frequency versus Damping diagram. Here the weighted mean values are presented as the center point of the confidence ellipsoid, estimated from the covariance of the modal frequency and damping ratio pair. In the diagrams presented in Figures 4 and 5, the ellipsoids present the 95% confidence bounds. Basically, they indicate that with a repetition of the test the modal parameters will be located with 95% confidence inside this ellipsoid.

Typically, these confidence ellipsoids are horizontally very thin, indicating that the modal frequencies are more accurately estimated compared to the damping ratios. If the confidence ellipsoids are “tilting” it indicates correlation between the modal frequency and the damping ratio, i.e. a large cross-covariance between the two estimates.

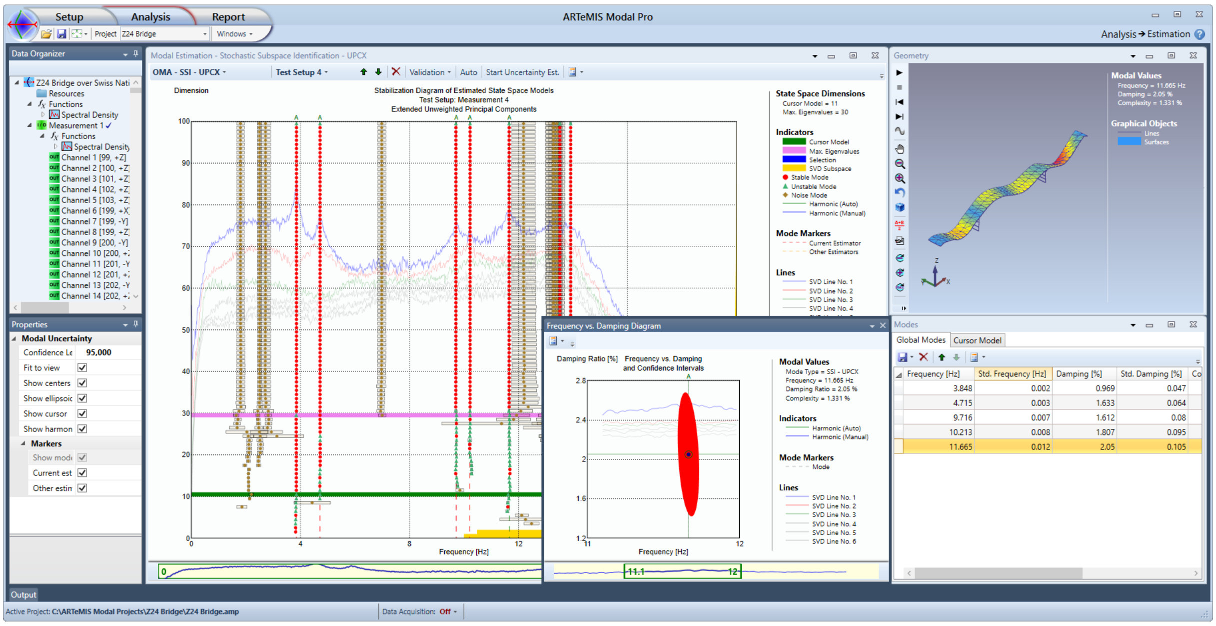

Figure 4: Stabilization diagram for Z24 Bridge with confidence ellipsoid of fifth mode.

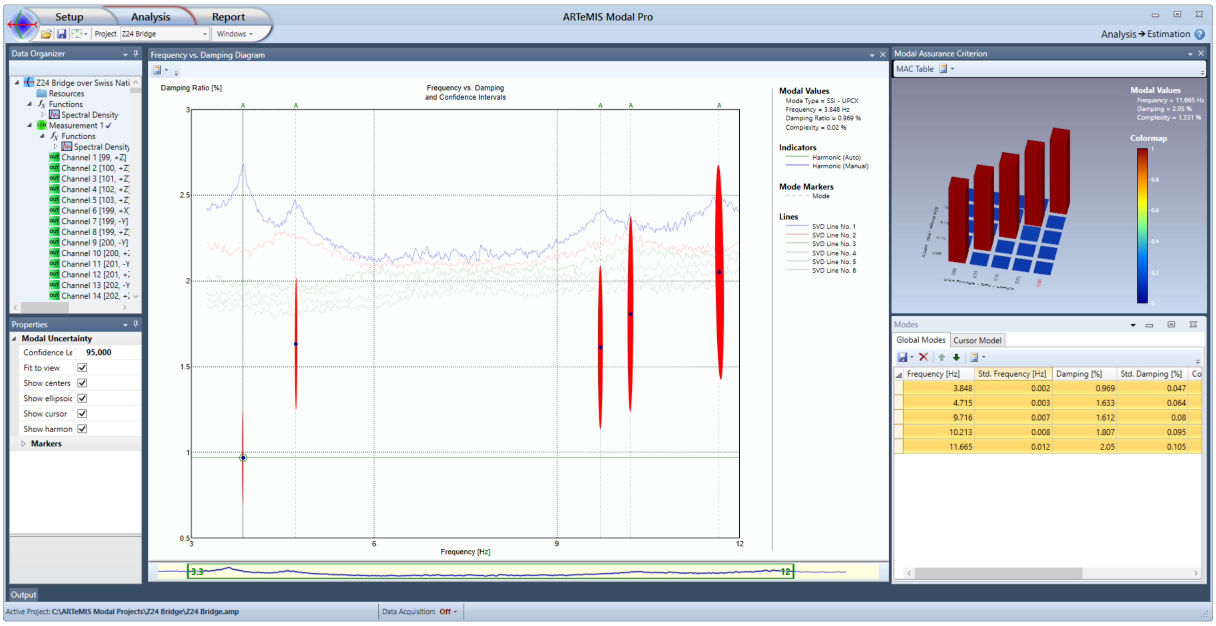

Figure 5: Confidence ellipsoids of first five modes.

6. CONCLUSION

In this paper, we have presented an efficient and automated method for the estimation of modal parameter uncertainties with the data-driven UPC subspace algorithm. The uncertainty is obtained on the same dataset as the underlying modal parameter estimate. We have proposed a procedure to take this uncertainty into account in the computation of a global mode estimate from the stabilization diagram, where the frequency and damping ratio estimates at the different model orders are weighted by their uncertainties. Finally, the uncertainty of the global mode estimate is computed. The method has been applied to the Z24 Bridge benchmark.

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