Contents

4 Equivalent Discrete-Time Structural Systems

In this chapter, it is analysed how to represent the combined continuous-time system derived in chapter 3 by discrete-time ARMAV models. In order to obtain such a discrete-time representation, it is necessary to sample the continuous-time combined system in some way. The sampled system will as such be a discrete-time equivalent to the continuous-time combined system. Sampling or discretization of the response of a stochastically excited continuous-time system can be performed in a number of ways, and no matter how it is performed information is lost as a consequence of the discretization. In principle, the difference between the discretization approaches is the way this loss is accounted for. Thus, any discrete model will only be an approximation of the continuous-time system and as such also of the true system.

There are at least four different approaches that can be applied to discretization of a continuous-time linear time-invariant system.

☞ *Approximation of differentials by difference equations.*
☞ *Approximation of the transfer function by pole-zero mapping techniques.*
☞ *Approximation of the transfer function by the zero-order hold-equivalence technique.*
☞ *Approximation of the system response by a covariance equivalence technique.*

The first approach approximates the differential equation of the combined system by difference equations, using one of the three approximation schemes; the forward rectangular rule, the backward rectangular rule, or the trapezoidal rule. The first rule is also known as Euler's rule, whereas the last sometimes is referred to as Tustin's rule. This discretization approach is described in Safak [97]. The second approach approximates the equivalent discrete-time system by matching the poles and zeroes of the transfer functions of the two systems, see Åström et al. [114]. In the third approach, the continuous-time excitation of the system is held constant by assuming a zero-order hold. The continuous-time system is then subjected to this discrete-time input and the result is a discrete-time output. The discrete-time transfer function can then be obtained as the ratio between the discrete-time input and output, see Middleton et al. [79]. However, all these techniques requires measured input and can as such not be applied to analysis of ambient excited structures. In the last approach, which will be adopted here, the equivalent discrete-time system, is approximated by requiring that the covariance function of the system response for a Gaussian white noise input coincides at all discrete time lags with that of the continuous-time system. The reason why this approach is adopted is that ambient excitation is unknown which makes the system response the only information available about the system. Assuming that this response is Gaussian distributed a covariance equivalent model will thus be exact at all discrete time steps and as such be the most accurate approximation approach.

For univariate second-order systems this approximation technique has been discussed by several authors, see Bartlett [13], Gersch et al. [27], Kozin et al. [68] and Pandit et al. [88]. The generalization to multivariate second-order systems has been considered in Andersen et al. [5]. However, as shown in the previous chapter the white noise excited combined continuous-time system is only of second order in special cases. Besides the stochastic excitation that affects the combined continuous-time system, the system will most certainly also be affected by different forms of disturbance. This disturbance is e.g. due to the approximation of the true system with a discrete-time model, or simply measurement noise. The presence of disturbance must also be accounted for by the equivalent discrete-time model. In this chapter, it is shown how to obtain a covariance equivalent discrete-time stochastic model to a multivariate continuous-system of arbitrary order. This covariance equivalence will be established for a noise-free system. However, it will also be shown how to account for the presence of noise in the system. The primary objective of this chapter though, is to serve as a basis for the development guidelines for the selection of an appropriate discrete-time model structure for use in practical system identification. Practical guidelines for selection of an appropriate model structure should be able to answer the following questions

- ☞ *What kind of model structure should be applied?*
- ☞ *What should the dimensions of this structure when noise is not present?*
- ☞ *What should the dimensions of this structure when noise is present?*

The first two questions are answered in the instance when the covariance equivalent discrete-time model of the noise-free combined continuous-time system is obtained. The last question can then be answered by applying the techniques developed in chapter 2. The actual formulation of the practical guidelines is given in section 5.7. In section 4.1, it is considered how to sample the continuous-time system and how to interpret covariance equivalence. Based on this knowledge, it is in section 4.2 shown how to obtain a covariance equivalent discrete-time model of the combined continuous-time system. In section 4.3, it is analysed how to make this model account for additional disturbance.

4.1 Sampling of the Combined Continuous-Time System

Assume that the state vector $\boldsymbol{x}(t_1)$ of the continuous-time state space system in (3.46) is known at some time t_1. The state vector $\boldsymbol{x}(t_2)$ at the time t_2 can then be determined from the solution, see e.g. Kailath [48]

$$\boldsymbol{x}(t_2) = e^{\boldsymbol{F}(t_2-t_1)}\boldsymbol{x}(t_1) + \int_{t_1}^{t_2} e^{\boldsymbol{F}(t_2-\tau)}\boldsymbol{B}\boldsymbol{w}(\tau)\,d\tau \;, \quad \boldsymbol{w}(t) \in NID(\boldsymbol{0},\boldsymbol{W}) \tag{4.1}$$

Let T be a constant sampling interval, and define the times $t_1 = kT$ and $t_2 = (k+1)T$, with k being an arbitrary integer. Inserting these definitions into (4.1) yields

$$\begin{aligned} x((k+1)T) &= e^{F((k+1)T-kT)}x(kT) + \int_{kT}^{(k+1)T} e^{F((k+1)T-\tau)}Bw(\tau)\,d\tau \\ &= e^{FT}x(kT) + \int_{kT}^{(k+1)T} e^{F((k+1)T-\tau)}Bw(\tau)\,d\tau \end{aligned} \tag{4.2}$$

The response $y(kT)$ is trivially obtained using the observation equation in (3.46) as

$$y(kT) = Cx(kT) \tag{4.3}$$

This discretization approach is called direct sampling since it corresponds to experimental sampling, see e.g. Wahlberg et al. [111]. At each discrete time instance a snapshot of the system is taken.

4.1.1 The Sampled Discrete-Time State Space System

Define the discrete time step t_k as $t_k = kT$. Further, define the transition matrix as

$$A = e^{FT} \tag{4.4}$$

and the input $\tilde{w}(t_k)$ as

$$\tilde{w}(t_k) = \int_{kT}^{(k+1)T} e^{F((k+1)T-\tau)}Bw(\tau)\,d\tau \tag{4.5}$$

The input is zero-mean and Gaussian distributed, since integration is a linear operation and $w(t)$ itself is zero-mean and Gaussian distributed. The integrations are non-overlapping, implying that $\tilde{w}(t_k)$, for different k, are statistically independent. The process $\tilde{w}(t_k)$ is therefore a discrete-time Gaussian white noise and can be completely described by the covariance matrix Ω, given by

$$\begin{aligned} \Omega &= E\left[\tilde{w}(t_k)\tilde{w}^T(t_k)\right] \\ &= \int_{(k-1)T}^{kT} e^{F(kT-\tau)}BWB^Te^{F^T(kT-\tau)}\,d\tau \\ &= \int_0^T e^{Ft}BWB^Te^{F^Tt}\,dt \end{aligned} \tag{4.6}$$

In (4.6) a variable substitution $t = kT\text{-}\tau$ reveals that Ω is independent of k and that $\tilde{\boldsymbol{w}}(t_k)$ therefore is stationary. Insert (4.4) and (4.5) into (4.2) to yield the discrete-time state space system

$$\begin{aligned} \boldsymbol{x}(t_{k+1}) &= \boldsymbol{A}\boldsymbol{x}(t_k) + \tilde{\boldsymbol{w}}(t_k)\ , \qquad \tilde{\boldsymbol{w}}(t_k) \in NID(\boldsymbol{0}, \boldsymbol{\Omega}) \\ \boldsymbol{y}(t_k) &= \boldsymbol{C}\boldsymbol{x}(t_k) \end{aligned} \tag{4.7}$$

As might be observed, this system resembles the state space representation defined in (2.1) without a direct term.

4.1.2 Covariance Equivalence

The key issue of discrete-time modelling of a continuous-time system is how to express the integral in (4.2). This is the major difference between the different discretization approaches mentioned at the beginning of this chapter. The dynamic properties of the system are the same no matter how the discretization has been obtained. This section concerns the covariance equivalence technique.

Since the input to the linear time-invariant combined continuous-time system (3.46) is stationary Gaussian white noise, the response will also be stationary and Gaussian distributed. This implies that the first and second-order moments of the response provides a complete statistical description of the system response. Without loss of generality, it will be assumed that the response process has zero-mean. This implies that the covariance function describes it completely. From theorem A.2, in appendix A, the covariance function $\boldsymbol{\Gamma}(\tau)$ of the response $\boldsymbol{y}(t)$ of the state space realization (3.46) is given by

$$\begin{aligned} \boldsymbol{\Pi}(\tau) &= e^{\boldsymbol{F}\tau}\boldsymbol{\Pi}(0) \\ \boldsymbol{\Gamma}(\tau) &= \boldsymbol{C}\boldsymbol{\Pi}(\tau)\boldsymbol{C}^T \end{aligned} \tag{4.8}$$

where $\boldsymbol{\Pi}(\tau)$ is the covariance at time lag τ of the state vector $\boldsymbol{x}(t)$. $\boldsymbol{\Pi}(0)$ is obtained as a positive definite solution of the Lyapunov equation

$$\boldsymbol{F}\boldsymbol{\Pi}(0) + \boldsymbol{\Pi}(0)\boldsymbol{F}^T = -\boldsymbol{B}\boldsymbol{W}\boldsymbol{B}^T \tag{4.9}$$

Let s be an arbitrary integer, and consider only discrete time lags $\tau = sT$. It then follows that (4.8) can be expressed in terms of the transition matrix as

$$\begin{aligned} \boldsymbol{\Pi}(sT) &= \boldsymbol{A}^s\boldsymbol{\Pi}(0) \\ \boldsymbol{\Gamma}(sT) &= \boldsymbol{C}\boldsymbol{\Pi}(sT)\boldsymbol{C}^T \end{aligned} \tag{4.10}$$

Define the covariance function of the discrete-time response $\boldsymbol{y}(t_k)$ as $\boldsymbol{\Sigma}(s)$. If this covariance function can be expressed explicitly in terms of the continuous-time covariance function at all discrete time lags, i.e.

$$\boldsymbol{\Sigma}(s) = \boldsymbol{\Gamma}(sT) \tag{4.11}$$

then the system response of the continuous-time and discrete-time systems is covariance equivalent for all discrete time steps. This is illustrated in figure 4.1.

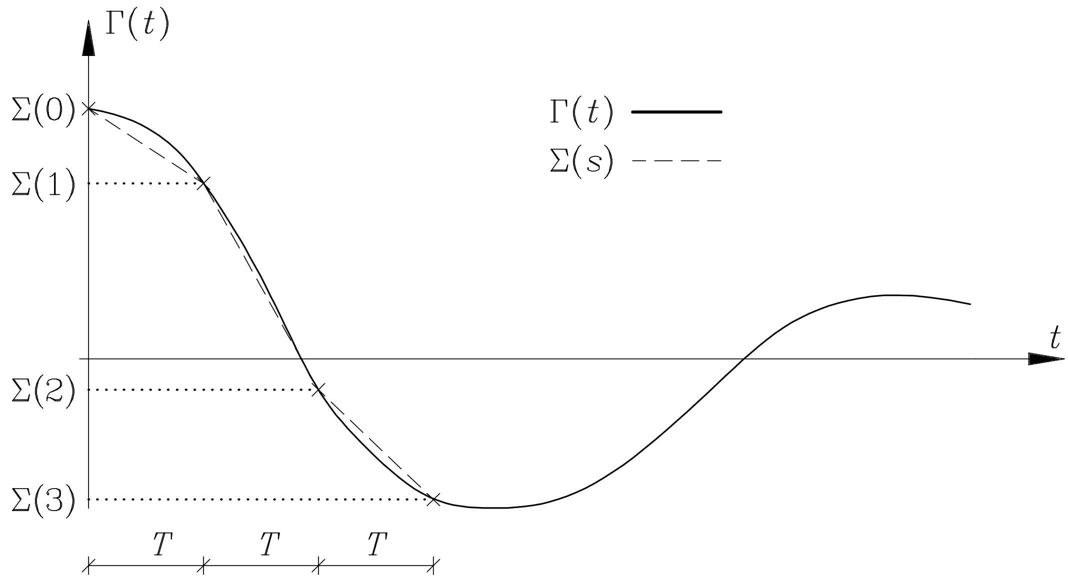

Figure 4.1: Covariance equivalence. $\boldsymbol{\Gamma}(t)$ is the continuous-time covariance function and $\boldsymbol{\Sigma}(s)$ is the discrete-time covariance function. These are related through the sampling interval T.

In the context of system identification, it should be noticed that due to the Wiener-Khinchine relation, see e.g. Bendat et al. [14], an estimated covariance equivalent discrete-time model provides an unbiased estimate of the spectral densities of the true system. The question is whether (4.7) is a covariance equivalent discrete-time representation of the combined continuous-time system (3.46) or not, i.e. is the relation (4.11) fulfilled or not? To answer this, multiply (4.7) from the right with $\boldsymbol{x}^T(t_{k-s})$ to yield

$$\boldsymbol{x}(t_{k+1})\boldsymbol{x}^T(t_{k-s}) = \boldsymbol{A}\boldsymbol{x}(t_k)\boldsymbol{x}^T(t_{k-s}) + \tilde{\boldsymbol{w}}(t_k)\boldsymbol{x}^T(t_{k-s}) \tag{4.12}$$

Taking the expectation on both sides results in the following covariance relation

$$\boldsymbol{\Pi}((s+1)T) = \boldsymbol{A}\boldsymbol{\Pi}(sT) \tag{4.13}$$

where the relation between the continuous and discrete time-instances has been stressed out. This equation can be solved by recursive substitutions of itself to yield

$$\boldsymbol{\Pi}(sT) = \boldsymbol{A}^s \boldsymbol{\Pi}(0) \tag{4.14}$$

where $\boldsymbol{\Pi}(0)$ is the zero-lag covariance obtained from (4.9). The covariance of the output $\boldsymbol{y}(t_k)$ is trivially obtained from

$$\begin{aligned} \boldsymbol{\Sigma}(s) &= E\left[\boldsymbol{y}(t_k)\boldsymbol{y}^T(t_{k-s})\right] \\ &= \boldsymbol{C}E\left[\boldsymbol{x}(t_k)\boldsymbol{x}^T(t_{k-s})\right]\boldsymbol{C}^T \\ &= \boldsymbol{C}\boldsymbol{\Pi}(sT)\boldsymbol{C}^T \end{aligned} \tag{4.15}$$

If (4.14) and (4.15) are compared with (4.10), it is seen that (4.11) is fulfilled and that (4.7) is a covariance equivalent discrete-time representation of (3.46).

4.2 Equivalent ARMAV Models - Without Noise Modelling

The objective of this section is to obtain a covariance equivalent ARMAV model of the noise-free combined continuous-time system (3.46) of the type presented in section 2.3. Assume that the dimension of the state matrix $\boldsymbol{F}$ is $m \times m$, and that the number of observed outputs is p. Since the dimensions of $\boldsymbol{A}$ and $\boldsymbol{F}$ are the same, this relation implies that the nth order continuous-time differential system is equivalent to an nth order difference equation system

$$\boldsymbol{y}(t_k) + \boldsymbol{A}_1\boldsymbol{y}(t_{k-1}) + \boldsymbol{A}_2\boldsymbol{y}(t_{k-2}) + \dots + \boldsymbol{A}_n\boldsymbol{y}(t_{k-n}) = \boldsymbol{z}(t_k) \tag{4.16}$$

with $n = \frac{m}{p}$ and $\boldsymbol{z}(t_k)$ being a fixed particular solution, see section 2.3. Since the order of the homogenous part of this discrete-time system is n, and since the observation matrix $\boldsymbol{C}$ remains unchanged during sampling, the $p \times p$ coefficient matrices $\boldsymbol{A}_i$ of this difference equation is obtained directly from (2.42) and (4.4) as

$$\left[\boldsymbol{A}_n \;\; \boldsymbol{A}_{n-1} \;\; . \;\; . \;\; \boldsymbol{A}_2 \;\; \boldsymbol{A}_1\right] = -\boldsymbol{C}\boldsymbol{A}^n\boldsymbol{Q}_o^{-1}(n) \tag{4.17}$$

with $\boldsymbol{Q}_o(n)$ being the observability matrix constructed from $\{\boldsymbol{A}, \boldsymbol{C}\}$.

Since the sampled combined continuous-time system (4.7) is driven by white noise $\tilde{\boldsymbol{w}}(t_k)$ it is possible to represent it by a discrete-time ARMAV model, according to theorem 2.2. This implies that the vector function $\boldsymbol{z}(t_k)$ is a moving average matrix polynomial. The purpose of this moving average is to secure that the output $\boldsymbol{y}(t_k)$ is stationary. The problem that remains to be solved is therefore the determination of the moving average, i.e. parameterize the function $\boldsymbol{z}(t_k)$ in such a way that (4.11) is satisfied. This parameterization is determined by the following theorem.

Theorem 4.1 - An Equivalent ARMAV(*n*,*n*-1) Model - Without Noise Modelling

The covariance equivalent discrete-time model to a directly sampled *n*th order *p*-variate continuous-time system, given by (3.46), can be represented by an ARMAV(*n*,*n*-1) model defined as

$$\begin{aligned} \boldsymbol{y}(t_k) + \boldsymbol{A}_1\boldsymbol{y}(t_{k-1}) + \boldsymbol{A}_2\boldsymbol{y}(t_{k-2}) + \ldots + \boldsymbol{A}_n\boldsymbol{y}(t_{k-n}) &= \\ \boldsymbol{B}_1\boldsymbol{u}(t_{k-1}) + \boldsymbol{B}_2\boldsymbol{u}(t_{k-2}) + \ldots + \boldsymbol{B}_n\boldsymbol{u}(t_{k-n}) & \end{aligned} \tag{4.18}$$

$$\boldsymbol{u}(t_k) \in NID(\boldsymbol{0}, \Delta)$$

Proof:

Consider the sampled solution (4.7) of (3.46). Since the auto-regressive coefficient matrices $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_n$ have been obtained from $\boldsymbol{A}$ by means of $\boldsymbol{C}$, it follows that (2.40) is satisfied. This fact is used in the following. Observe from (4.2) and (4.7) that $\boldsymbol{y}(t_{k+p})$ can be written as

$$\begin{aligned} \boldsymbol{y}(t_{k+p}) &= \boldsymbol{C}\boldsymbol{A}^p\boldsymbol{x}(t_k) + \sum_{j=1}^{p} \boldsymbol{C}\boldsymbol{A}^{p-j}\tilde{\boldsymbol{w}}(t_{k+j-1}) \\ \tilde{\boldsymbol{w}}(t_k) &\in NID(\boldsymbol{0}, \Omega) \end{aligned} \tag{4.19}$$

by making recursive substitutions of (4.7) itself. From the observation equation of (4.7) and (4.19), construct the following set of equations

$$\begin{bmatrix} \boldsymbol{y}(t_{k-n}) \\ \boldsymbol{y}(t_{k-n+1}) \\ . \\ . \\ \boldsymbol{y}(t_{k-1}) \\ \boldsymbol{y}(t_k) \end{bmatrix} = \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{C}\boldsymbol{A} \\ . \\ . \\ \boldsymbol{C}\boldsymbol{A}^{n-1} \\ \boldsymbol{C}\boldsymbol{A}^{n} \end{bmatrix} \boldsymbol{x}(t_{k-n}) + \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & . & . & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{C} & \boldsymbol{0} & . & . & \boldsymbol{0} & \boldsymbol{0} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \boldsymbol{C}\boldsymbol{A}^{n-2} & \boldsymbol{C}\boldsymbol{A}^{n-3} & . & . & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{C}\boldsymbol{A}^{n-1} & \boldsymbol{C}\boldsymbol{A}^{n-2} & . & . & \boldsymbol{C} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{w}}(t_{k-n}) \\ \tilde{\boldsymbol{w}}(t_{k-n+1}) \\ . \\ . \\ \tilde{\boldsymbol{w}}(t_{k-1}) \\ \tilde{\boldsymbol{w}}(t_k) \end{bmatrix} \tag{4.20}$$

which can compactly be written as

$$\boldsymbol{Y}(t_{k-n},t_k) \;=\; \boldsymbol{Q}_o(n+1)\boldsymbol{x}(t_k) \;+\; \boldsymbol{T}(n+1)\tilde{\boldsymbol{W}}(t_{k-n},t_k) \tag{4.21}$$

Now introduce the result of (4.17), which is a matrix containing the auto-regressive coefficient matrices,

$$\boldsymbol{P} \;=\; \begin{bmatrix} \boldsymbol{A}_n & \boldsymbol{A}_{n-1} & . & . & \boldsymbol{A}_1 & \boldsymbol{I} \end{bmatrix} \tag{4.22}$$

and multiply (4.21) from the left by (4.22) to yield

$$\begin{aligned} \boldsymbol{P}\boldsymbol{Y}(t_{k-n},t_k) \;&=\; \boldsymbol{P}\boldsymbol{Q}_o(n+1)\boldsymbol{x}(t_k) \;+\; \boldsymbol{P}\boldsymbol{T}(n+1)\tilde{\boldsymbol{W}}(t_{k-n},t_k) \\ &=\; \boldsymbol{P}\boldsymbol{T}(n+1)\tilde{\boldsymbol{W}}(t_{k-n},t_k) \\ &=\; \tilde{\boldsymbol{B}}_1\tilde{\boldsymbol{w}}(t_{k-1}) \;+\; \tilde{\boldsymbol{B}}_2\tilde{\boldsymbol{w}}(t_{k-2}) \;+\; \dots \;+\; \tilde{\boldsymbol{B}}_n\tilde{\boldsymbol{w}}(t_{k-n}) \\ &=\; \boldsymbol{z}(t_k) \end{aligned} \tag{4.23}$$

$$\tilde{\boldsymbol{B}}_j \;=\; \boldsymbol{C}\boldsymbol{A}^{j-1} \;+\; \boldsymbol{A}_1\boldsymbol{C}\boldsymbol{A}^{j-2} \;+\; \dots \;+\; \boldsymbol{A}_{j-1}\boldsymbol{C} \;, \qquad 0<j\le n$$

The left hand-side of (4.23) is exactly the auto-regressive part of (4.18), and the right-hand term $\boldsymbol{z}(t_k)$ is a matrix polynomial in $\tilde{\boldsymbol{w}}(t_k)$ with $\tilde{\boldsymbol{B}}_j$ being $p \times m$ coefficient matrices. Since the process $\tilde{\boldsymbol{w}}(t_k)$ is a discrete-time Gaussian white noise $\boldsymbol{z}(t_k)$ is certainly a moving average. Also, since the first p rows and the last m columns of $\boldsymbol{T}(n+1)$ are all zero, the moving average coefficient matrix $\tilde{\boldsymbol{B}}_0$ associated with $\tilde{\boldsymbol{w}}(t_k)$ is zero. The reason is that the sampled state space system (4.7) does not have a direct term in the observation equation. This implies that the moving average matrix polynomial in reality is only of the order n-1, and that the complete ARMAV(n,n-1) model is given by (4.18). However, the dimension of $\tilde{\boldsymbol{w}}(t_k)$ is different from the dimension of $\boldsymbol{y}(t_k)$. This implies that (4.22) at the present moment does not have the standard form of the multivariate ARMAV model of theorem 2.2. It should be noticed though, that the continuous-time white noise $\boldsymbol{w}(t)$ and $\boldsymbol{z}(t_k)$ both are p-variate vectors. This suggests that the moving average $\boldsymbol{z}(t_k)$ can be represented by a matrix polynomial of the following kind

$$\boldsymbol{z}(t_k) \;=\; \boldsymbol{B}_1\boldsymbol{u}(t_{k-1}) \;+\; \boldsymbol{B}_2\boldsymbol{u}(t_{k-2}) \;+\; \dots \;+\; \boldsymbol{B}_n\boldsymbol{u}(t_{k-n}) \tag{4.24}$$

where $\boldsymbol{B}_i$ are $p \times p$ coefficient matrices, and $\boldsymbol{u}(t_k)$ is an p-variate discrete-time zero-mean stationary Gaussian white noise process. This process is fully described by its $p \times p$ covariance matrix $\boldsymbol{\Delta}$. Since $\boldsymbol{z}(t_k)$ is a Gaussian distributed process with zero mean the only requirement for (4.24) to be statistically equivalent to (4.23) is that the covariance function of $\boldsymbol{z}(t_k)$ is the same in both cases. From (4.23) the covariance function is given by

$$E\Big[\boldsymbol{z}(t_k)\boldsymbol{z}^T(t_{k+j})\Big] \;=\; \begin{cases} \tilde{\boldsymbol{B}}_1\Omega\tilde{\boldsymbol{B}}_1^T \;+\; \dots \;+\; \tilde{\boldsymbol{B}}_n\Omega\tilde{\boldsymbol{B}}_n^T & ,\; j=0 \\ \tilde{\boldsymbol{B}}_1\Omega\tilde{\boldsymbol{B}}_{j+1} \;+\; \dots \;+\; \tilde{\boldsymbol{B}}_{n-j}\Omega\tilde{\boldsymbol{B}}_n^T & ,\; 0<j<n \\ \boldsymbol{0} & ,\; j\ge n \end{cases} \tag{4.25}$$

This covariance function can be represented in a similar way in terms of (4.24) to yield

$$E\left[\boldsymbol{z}(t_k)\boldsymbol{z}^T(t_{k+j})\right] = \begin{cases} \boldsymbol{B}_1\boldsymbol{\Delta}\boldsymbol{B}_1^T + \ldots + \boldsymbol{B}_n\boldsymbol{\Delta}\boldsymbol{B}_n^T & , \quad j=0 \\ \boldsymbol{B}_1\boldsymbol{\Delta}\boldsymbol{B}_{j+1} + \ldots + \boldsymbol{B}_{n-j}\boldsymbol{\Delta}\boldsymbol{B}_n^T & , \quad 0<j<n \\ \boldsymbol{0} & , \quad j \geq n \end{cases} \tag{4.26}$$

In both cases there are one auto-covariance term and n-1 non-trivial cross-covariance terms that must be fulfilled. Of course similar expressions of the cross-covariances in (4.25) and (4.26) for $j < 0$ will also exist. However, in the present context such expressions are immaterial since they will not provide any additional information. Equating (4.26) and (4.25), the following n equations are obtained

$$\boldsymbol{B}_1\boldsymbol{\Delta}\boldsymbol{B}_{j+1} + \ldots + \boldsymbol{B}_{n-j}\boldsymbol{\Delta}\boldsymbol{B}_n^T = \tilde{\boldsymbol{B}}_1\boldsymbol{\Omega}\tilde{\boldsymbol{B}}_{j+1} + \ldots + \tilde{\boldsymbol{B}}_{n-j}\boldsymbol{\Omega}\tilde{\boldsymbol{B}}_n^T = \boldsymbol{C}_j \tag{4.27}$$

where $\boldsymbol{C}_j$, for j = 0 to n-1, are known $p \times p$ matrices.

The total number of unknown matrices in this equation system is $n+1$, since there are n unknown coefficient matrices $\boldsymbol{B}_j$ and one unknown covariance matrix $\boldsymbol{\Delta}$. All unknown matrices are of the dimension $p \times p$. On the other hand, only the n matrices $\boldsymbol{C}_j$ are known, which implies that there are too many unknowns. This problem can, however, be eliminated by selecting the elements of one of the unknown matrices. A sensible choice could be to select the elements of the first coefficient matrix as $\boldsymbol{B}_1 = \boldsymbol{I}$. The result is then n p-variate equations and n p-variate unknowns. This implies that it must be possible to determine a set of coefficient matrices $\boldsymbol{B}_j$, for j = 2 to n, and a covariance matrix $\boldsymbol{\Delta}$, that satisfies (4.27). There will certainly be many solutions, due to the dimension p of the n unknown matrices in (4.27), and the actual number of solutions depends both on p and n. This also explains why it is impossible to determine a unique moving average of the standard ARMAV model in the context of system identification.
□

As seen a theoretical relation between the combined continuous-time system and the discrete-time covariance equivalent ARMAV model exist. This ARMAV(n,n-1) model can be represented by the following state space system

$$\begin{aligned} \boldsymbol{x}(t_{k+1}) &= \boldsymbol{A}\boldsymbol{x}(t_k) + \boldsymbol{B}\boldsymbol{u}(t_k) \ , \quad \boldsymbol{u}(t_k) \in NID(\boldsymbol{0},\boldsymbol{\Delta}) \\ \boldsymbol{y}(t_k) &= \boldsymbol{C}\boldsymbol{x}(t_k) \end{aligned} \tag{4.28}$$

since it is the lack of a direct term in the observation equation that reduces the moving average order. The actual state space realization of the ARMAV(n,n-1) could e.g. be of the observability canonical form shown in theorem 2.4. It is difficult in the general case to calculate the moving average coefficients explicitly. However, in system identification this is unimportant. However, there might be applications, such as simulation, where it would be desirable to have an analytical relationship between the continuous-time system and a covariance equivalent ARMAV model. Such a relationship can be established if the combined continuous-time system (3.46) is of the second order. This will be considered in the following special case.

4.2.1 A Special Case: An ARMAV(2,1) Model

Theorem 4.1 does not provide any explicit scheme for the determination of the moving average coefficient matrices and the corresponding covariance matrix of the Gaussian white noise input. This section describes how a covariance equivalent ARMAV(2,1) model can be determined explicitly. The continuous-time second-order system was introduced in (3.1) in section 3.1.1 as

$$\ddot{\boldsymbol{z}}(t) + \boldsymbol{M}^{-1}\boldsymbol{C}\dot{\boldsymbol{z}}(t) + \boldsymbol{M}^{-1}\boldsymbol{K}\boldsymbol{z}(t) = \boldsymbol{M}^{-1}\boldsymbol{w}(t) \ , \quad \boldsymbol{w}(t) \in NID(\boldsymbol{0}, \boldsymbol{W}) \tag{4.29}$$

Here $\boldsymbol{w}(t)$ is a zero-mean continuous-time Gaussian white noise, completely described by the intensity matrix $\boldsymbol{W}$. This system is obtained from (3.39), for $n = 2$, by the following definition of the coefficient matrices $\{\boldsymbol{C}_{z,0}, \boldsymbol{C}_{z,1}, \boldsymbol{C}_{w,0}\}$

$$\boldsymbol{C}_{z,0} = \boldsymbol{M}^{-1}\boldsymbol{K} \ , \quad \boldsymbol{C}_{z,1} = \boldsymbol{M}^{-1}\boldsymbol{C} \ , \quad \boldsymbol{C}_{w,0} = \boldsymbol{M}^{-1} \tag{4.30}$$

Assume that the outputs of the system are the displacements of all mass points in the model. In this case $\{\boldsymbol{F}, \boldsymbol{B}, \boldsymbol{C}\}$ of the state space realization (3.46) are given by

$$\boldsymbol{F} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{M}^{-1}\boldsymbol{K} & -\boldsymbol{M}^{-1}\boldsymbol{C} \end{bmatrix} , \quad \boldsymbol{B} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{M}^{-1} \end{bmatrix} , \quad \boldsymbol{C} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \end{bmatrix} \tag{4.31}$$

By choosing a sampling interval T the transition matrix $\boldsymbol{A}$ can be calculated from (4.4). From the following theorem, it is then possible to calculate the covariance equivalent ARMAV(2,1) model explicitly.

Theorem 4.2 - A Covariance Equivalent ARMAV(2,1) Model.

Consider the Gaussian white noise excited second-order combined continuous-time system (4.31). A covariance equivalent ARMAV(2,1) model to this system is defined as

$$\begin{aligned} \boldsymbol{y}(t_k) + \boldsymbol{A}_1\boldsymbol{y}(t_{k-1}) + \boldsymbol{A}_2\boldsymbol{y}(t_{k-2}) &= \\ \boldsymbol{u}(t_{k-1}) + \boldsymbol{B}_2\boldsymbol{u}(t_{k-2}) \ , &\quad \boldsymbol{u}(t_k) \in NID(\boldsymbol{0}, \boldsymbol{\Delta}) \end{aligned} \tag{4.32}$$

Based on the transition matrix $\boldsymbol{A}$, given in (4.4), and the sampling interval T, the auto-regressive coefficient matrices are calculated by

$$\begin{bmatrix} \boldsymbol{A}_2 & \boldsymbol{A}_1 \end{bmatrix} = -\boldsymbol{C}\boldsymbol{A}^2 \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{C}\boldsymbol{A} \end{bmatrix}^{-1} \tag{4.33}$$

The moving average coefficient matrix is obtained from the solution of the second order matrix polynomial

$$\boldsymbol{B}_2^2 - \boldsymbol{B}_2\left(\boldsymbol{K}_0 + \boldsymbol{K}_1\boldsymbol{A}_1^T\right)\boldsymbol{K}_1^{-T} + \boldsymbol{K}_1\boldsymbol{K}_1^{-T} = \boldsymbol{0} \tag{4.34}$$

with $\boldsymbol{K}_0$ and $\boldsymbol{K}_1$ defined as

$$\begin{aligned} \boldsymbol{K}_0 &= \boldsymbol{\Gamma}(0) + \boldsymbol{A}_1\boldsymbol{\Gamma}^T(T) + \boldsymbol{A}_2\boldsymbol{\Gamma}^T(2T) \\ \boldsymbol{K}_1 &= \boldsymbol{\Gamma}(T) + \boldsymbol{A}_1\boldsymbol{\Gamma}(0) + \boldsymbol{A}_2\boldsymbol{\Gamma}^T(T) \end{aligned} \tag{4.35}$$

$\boldsymbol{\Gamma}(0)$, $\boldsymbol{\Gamma}(T)$ and $\boldsymbol{\Gamma}(2T)$ are obtained from (4.8). The solution of (4.34) is given in appendix B, and the covariance matrix $\boldsymbol{\Delta}$ of the discrete-time Gaussian white noise $\boldsymbol{u}(t_k)$ is obtained from

$$\boldsymbol{\Delta} = \boldsymbol{B}_2^{-1}\boldsymbol{K}_1 \tag{4.36}$$

Since (4.34) is a matrix polynomial, whose coefficient matrices are of the dimension $p \times p$, and since the solution matrix $\boldsymbol{B}_2$ also has the dimension $p \times p$, the number of independent solutions is $\binom{2p}{p}$. In the univariate case this number is two.

Proof:

Since the continuous-time system is of the second order, the auto-regressive order will also be two. Assuming that the continuous-time state space system is minimal, the two auto-regressive coefficient matrices $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are calculated directly from (4.17) by choosing the sampling interval T. Following theorem 4.1, covariance equivalence with a continuous-time second-order system can be obtained by the following ARMAV model

$$\begin{aligned} \boldsymbol{y}(t_k) + \boldsymbol{A}_1\boldsymbol{y}(t_{k-1}) + \boldsymbol{A}_2\boldsymbol{y}(t_{k-2}) = \\ \boldsymbol{B}_1\boldsymbol{u}(t_{k-1}) + \boldsymbol{B}_2\boldsymbol{u}(t_{k-2}) \;, \quad \boldsymbol{u}(t_k) \in NID(\boldsymbol{0},\boldsymbol{\Delta}) \end{aligned} \tag{4.37}$$

with $\boldsymbol{B}_1 = \boldsymbol{I}$. Consider the following implicit formulation of the discrete-time covariance function of an ARMAV(2,1) model, see e.g. Pandit et al. [88]

$$\begin{aligned} \boldsymbol{\Sigma}(k) + \boldsymbol{A}_1\boldsymbol{\Sigma}(k-1) + \boldsymbol{A}_2\boldsymbol{\Sigma}(k-2) = \\ \boldsymbol{\Delta}\boldsymbol{h}^T(1-k) + \boldsymbol{B}_2\boldsymbol{\Delta}\boldsymbol{h}^T(2-k) \end{aligned} \tag{4.38}$$

and insert $\boldsymbol{\Gamma}(kT)$ of the continuous-time system instead of $\boldsymbol{\Sigma}(k)$, to yield

$$\begin{aligned} \boldsymbol{\Gamma}(kT) + \boldsymbol{A}_1\boldsymbol{\Gamma}((k-1)T) + \boldsymbol{A}_2\boldsymbol{\Gamma}((k-2)T) = \\ \boldsymbol{\Delta}\boldsymbol{h}^T(1-k) + \boldsymbol{B}_2\boldsymbol{\Delta}\boldsymbol{h}^T(2-k) \end{aligned} \tag{4.39}$$

For simplicity introduce the following two $p \times p$ matrices

$$\begin{aligned} \boldsymbol{K}_0 &= \Gamma(0) + \boldsymbol{A}_1\Gamma^T(T) + \boldsymbol{A}_2\Gamma^T(2T) \\ \boldsymbol{K}_1 &= \Gamma(T) + \boldsymbol{A}_1\Gamma(0) + \boldsymbol{A}_2\Gamma^T(T) \end{aligned} \tag{4.40}$$

which are equal to the left-hand side of (4.39) by setting k equal to 0 and 1. Notice that

$$\begin{aligned} &\boldsymbol{h}(0) = \boldsymbol{0} \\ &\boldsymbol{h}(1) + \boldsymbol{A}_1\boldsymbol{h}(0) = \boldsymbol{I} \Leftrightarrow \boldsymbol{h}(1) = \boldsymbol{I} \\ &\boldsymbol{h}(2) + \boldsymbol{A}_1\boldsymbol{h}(1) + \boldsymbol{A}_2\boldsymbol{h}(0) = \boldsymbol{B}_2 \Leftrightarrow \boldsymbol{h}(2) = \boldsymbol{B}_2 - \boldsymbol{A}_1 \end{aligned} \tag{4.41}$$

which implies that (4.39) can be written as

$$\begin{aligned} \boldsymbol{K}_0 &= \Delta + \boldsymbol{B}_2\Delta\left(\boldsymbol{B}_2 - \boldsymbol{A}_1\right)^T \\ \boldsymbol{K}_1 &= \boldsymbol{B}_2\Delta \end{aligned} \tag{4.42}$$

By replacing Δ with $\boldsymbol{B}_2^{-1}\boldsymbol{K}_1$, obtained from the second equation of (4.42), in the first equation of (4.42), the following relation is obtained

$$\boldsymbol{K}_0 = \boldsymbol{B}_2^{-1}\boldsymbol{K}_1 + \boldsymbol{K}_1\left(\boldsymbol{B}_2 - \boldsymbol{A}_1\right)^T \tag{4.43}$$

from which $\boldsymbol{B}_2$ can be calculated. By rearrangement, the second-order matrix polynomial in (4.34) is obtained. The solution of this polynomial is given in appendix B. Based on $\boldsymbol{B}_2$ the covariance matrix Δ is determined from (4.42) or equivalently from (4.36). This approach, which is especially designed for ARMAV(2,1) models, is based on Andersen et al [5]. □

The covariance equivalent ARMAV(2,1) model can equivalently be represented by a state space realization with the state matrix $\boldsymbol{A}$, the input matrix $\boldsymbol{B}$, and the observation matrix $\boldsymbol{C}$ defined as

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{A}_2 & -\boldsymbol{A}_1 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{B}_2 - \boldsymbol{A}_1 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \end{bmatrix} \tag{4.44}$$

This definition follows directly from theorem 2.4. Notice, that since the state vector of the continuous-time system only includes the displacement and the velocity, it is only possible to construct a covariance equivalent discrete-time model that includes a linear combination of these.

4.2.2 Example 4.1: An ARMA(2,1) Model

By using the procedure described in the previous section, it is possible to obtain a covariance equivalent ARMA(2,1) model of the univariate second-order white noise excited structural system in example 3.1, section 3.3.2. The white noise excited continuous-time system is defined as, see (3.51)

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{F}\boldsymbol{x}(t) + \boldsymbol{B}w(t) \;, \qquad w(t) \in NID(0,W) \\ y(t) &= \boldsymbol{C}\boldsymbol{x}(t) \end{aligned} \tag{4.45}$$

With the matrix triple {$\boldsymbol{F}$, $\boldsymbol{B}$, $\boldsymbol{C}$} given by

$$\boldsymbol{F} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{4.46}$$

Assume that the eigenvalues {λ_1, λ_2} of $\boldsymbol{F}$ are distinct and stable, then the discrete-time eigenvalues {μ_1, μ_2 } are defined by the relation $\mu = e^{\lambda T}$. Since the eigenvectors of $\boldsymbol{F}$, defined as

$$\boldsymbol{\Psi} = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \tag{4.47}$$

diagonalize $\boldsymbol{F}$, see section 3.4.4, they will also diagonalize $e^{\boldsymbol{F}T}$, implying that

$$\begin{aligned} e^{\boldsymbol{F}T} &= \frac{1}{\lambda_2 - \lambda_1} \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 T} & 0 \\ 0 & e^{\lambda_2 T} \end{bmatrix} \begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix} \\ &= \frac{1}{\lambda_2 - \lambda_1} \begin{bmatrix} \lambda_2\mu_1 - \lambda_1\mu_2 & \mu_2 - \mu_1 \\ \lambda_1\lambda_2(\mu_1 - \mu_2) & \lambda_2\mu_2 - \lambda_1\mu_1 \end{bmatrix} \\ &= \boldsymbol{A} \end{aligned} \tag{4.48}$$

From {$\boldsymbol{A}$, $\boldsymbol{C}$} the observability matrix and its inverse are given by

$$\begin{aligned} \boldsymbol{Q}_o(2) &= \begin{bmatrix} 1 & 0 \\ \dfrac{\lambda_2\mu_1 - \lambda_1\mu_2}{\lambda_2 - \lambda_1} & \dfrac{\mu_2 - \mu_1}{\lambda_2 - \lambda_1} \end{bmatrix} \\ \boldsymbol{Q}_o^{-1}(2) &= \begin{bmatrix} 1 & 0 \\ \dfrac{\lambda_1\mu_2 - \lambda_2\mu_1}{\mu_2 - \mu_1} & \dfrac{\lambda_2 - \lambda_1}{\mu_2 - \mu_1} \end{bmatrix} \end{aligned} \tag{4.49}$$

and the row vector $\boldsymbol{CA}^2$ by

$$\boldsymbol{CA}^2 = \left[\frac{\lambda_2^2\mu_1^2 + \lambda_1^2\mu_2^2 - \lambda_1\lambda_2\left(\mu_1^2+\mu_2^2\right)}{\left(\lambda_2-\lambda_1\right)^2} \quad \frac{\left(\mu_2-\mu_1\right)\left(\mu_1+\mu_2\right)\left(\lambda_2-\lambda_1\right)}{\left(\lambda_2-\lambda_1\right)^2} \right] \tag{4.50}$$

implying that the two auto-regressive coefficients A_1 and A_2 are obtained from (4.17) as

$$\left[A_2 \quad A_1\right] = -\boldsymbol{CA}^2\boldsymbol{Q}_o^{-1} = \left[\mu_1\mu_2 \qquad -\mu_1-\mu_2\right] \tag{4.51}$$

The moving average parameter B_1 is determined from theorem 4.2. Calculate the covariances $\Gamma(0)$, $\Gamma(T)$ and $\Gamma(2T)$. These can be obtained from (4.8) in section 4.1.2. However, in the univariate case it is more straightforward to use the following relation, see Pandit et al. [88]

$$\Gamma(kT) = \frac{W}{2\lambda_1\lambda_2\left(\lambda_1^2-\lambda_2^2\right)}\left(\lambda_2\mu_1^k - \lambda_1\mu_2^k\right) \tag{4.52}$$

where W is the intensity of the univariate continuous-time Gaussian white noise $w(t)$. Based on the variances and the auto-regressive parameters the constants K_0 and K_1 in (4.40) in section 4.2.1 are calculated as

$$\begin{aligned} K_0 &= \frac{W\left(\lambda_1-\lambda_2-\left(\mu_1+\mu_2\right)\left(\lambda_2\mu_1-\lambda_1\mu_2\right)+\mu_1\mu_2\left(\lambda_2\mu_1^2-\lambda_1\mu_2^2\right)\right)}{2\lambda_1\lambda_2\left(\lambda_1^2-\lambda_2^2\right)} \\ K_1 &= \frac{W\left(\lambda_2\mu_1-\lambda_1\mu_2-\left(\mu_1+\mu_2\right)\left(\lambda_2-\lambda_1\right)+\mu_1\mu_2\left(\lambda_2\mu_1-\lambda_1\mu_2\right)\right)}{2\lambda_1\lambda_2\left(\lambda_1^2-\lambda_2^2\right)} \end{aligned} \tag{4.53}$$

and B_2 can then be determined as one of the solutions of the second-order polynomial

$$B_2^2 - CB_2 + 1 = 0 \tag{4.54}$$

with C defined as

$$\begin{aligned} C &= \frac{K_0}{K_1} + A_1 \\ &= \frac{\mu_1\mu_2\left(\lambda_1(1-\mu_2^2)-\lambda_2(1-\mu_1^2)\right)+\lambda_1\left(1+\mu_2^2\right)-\lambda_2\left(1+\mu_1^2\right)}{\mu_1\mu_2\left(\lambda_2\mu_1-\lambda_1\mu_2\right)+\lambda_1\mu_1-\lambda_2\mu_2} \\ &\quad - \mu_1 - \mu_2 \end{aligned} \tag{4.55}$$

to yield

$$B_1 = 0.5C \pm \sqrt{0.25C^2 - 1} \tag{4.56}$$

The corresponding variances Δ of the discrete-time Gaussian white noise process $u(t_k)$ is then given by

$$\Delta = \frac{W\Big(\lambda_1 - \lambda_2 - (\mu_1 + \mu_2)(\lambda_2\mu_1 - \lambda_1\mu_2) + \mu_1\mu_2\big(\lambda_2\mu_1^2 - \lambda_1\mu_2^2\big)\Big)}{2B_1\lambda_1\lambda_2\big(\lambda_1^2 - \lambda_2^2\big)} \tag{4.57}$$

□

4.3 Equivalent ARMAV Models - With Noise Modelling

The purpose of this section is to investigate what happens when sampled response is affected by measurement noise, and when the combined mathematical model of the structural system and the excitation do not fully match what is experienced in real-life structures exposed to ambient excitation. This inaccuracy will in the following be characterized as process noise. In the previous section the covariance equivalent ARMAV(n,n-1) model was derived, and it was shown in (4.28) that it could be represented in state space as

$$\begin{aligned} \boldsymbol{x}(t_{k+1}) &= \boldsymbol{A}\boldsymbol{x}(t_k) + \boldsymbol{B}\boldsymbol{u}(t_k) \ , \quad \boldsymbol{u}(t_k) \in NID(\boldsymbol{0},\boldsymbol{\Delta}) \\ \boldsymbol{y}(t_k) &= \boldsymbol{C}\boldsymbol{x}(t_k) \end{aligned} \tag{4.58}$$

The actual state space realization could e.g. be of the observability canonical form shown in theorem 2.4. In general, if noise is present, both state and observation equation will be affected as shown in section 2.2.1. Following (2.15), the presence of noise in the observations and in the system, changes (4.58) to

$$\begin{aligned} \boldsymbol{x}(t_{k+1}) &= \boldsymbol{A}\boldsymbol{x}(t_k) + \boldsymbol{B}\boldsymbol{u}(t_k) + \boldsymbol{w}(t_k) \\ \boldsymbol{y}(t_k) &= \boldsymbol{C}\boldsymbol{x}(t_k) + \boldsymbol{v}(t_k) \end{aligned} \tag{4.59}$$

where $\boldsymbol{w}(t_k)$ is the process noise that incorporates the system inaccuracies into the state of the system. $\boldsymbol{v}(t_k)$ is the measurement noise that describes the inaccuracy between the response of the modelled system and the measured system response $\boldsymbol{y}(t_k)$. The noise processes $\boldsymbol{w}(t_k)$ and $\boldsymbol{v}(t_k)$ are both assumed to be zero-mean Gaussian white noise, completely described by their covariance matrices $\boldsymbol{Q}$, $\boldsymbol{R}$, and $\boldsymbol{S}$, see section 2.2.1. The question is:

☞ *Does the presence of noise affect the external description of the discrete-time system?*

The answer is in general yes. The presence of noise will extend the moving average part.

In section 2.2, it was shown how to predict the system response of a noise-contaminated stochastic state space system. This prediction could be obtained using the steady-state Kalman filter. On the basis of the Kalman filter the innovation state space system was formulated as

$$
\begin{aligned}
\hat{\boldsymbol{x}}(t_{k+1}|t_k) &= \boldsymbol{A}\hat{\boldsymbol{x}}(t_k|t_{k-1}) + \boldsymbol{K}\boldsymbol{e}(t_k) \ , \quad \boldsymbol{e}(t_k) \in NID(\boldsymbol{0},\boldsymbol{\Lambda}) \\
\boldsymbol{y}(t_k) &= \boldsymbol{C}\hat{\boldsymbol{x}}(t_k|t_{k-1}) + \boldsymbol{e}(t_k)
\end{aligned}
\tag{4.60}
$$

where the innovation process $\boldsymbol{e}(t_k)$ is an equivalent Gaussian white noise process, that incorporates the actual stochastic excitation and the disturbance. The structure of the innovation state space system is seen to include a direct term of the innovations in the observation equation. The lack of this term was the reason why the order of the moving average of the covariance equivalent ARMAV model was n-1. Now, in the general case where the direct term is present it follows from theorem 2.3 that the ARMAV model which corresponds to (4.60) will have the form

$$
\begin{aligned}
&\boldsymbol{y}(t_k) + \boldsymbol{A}_1\boldsymbol{y}(t_{k-1}) + \boldsymbol{A}_2\boldsymbol{y}(t_{k-2}) + \ldots + \boldsymbol{A}_n\boldsymbol{y}(t_{k-n}) = \\
&\qquad \boldsymbol{e}(t_k) + \boldsymbol{C}_1\boldsymbol{e}(t_{k-1}) + \ldots + \boldsymbol{C}_n\boldsymbol{e}(t_{k-n}) \\
&\boldsymbol{e}(t_k) \in NID(\boldsymbol{0},\boldsymbol{\Lambda})
\end{aligned}
\tag{4.61}
$$

According to theorem 2.3 the moving average coefficient matrices is obtained from (2.54) as

$$
\begin{bmatrix} \boldsymbol{C}_n & \boldsymbol{C}_{n-1} & . & . & \boldsymbol{C}_1 & \boldsymbol{I} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_n & \boldsymbol{A}_{n-1} & . & . & \boldsymbol{A}_1 & \boldsymbol{I} \end{bmatrix} \boldsymbol{T}(n+1) \tag{4.62}
$$

with $\boldsymbol{T}(n+1)$ defined as

$$
\boldsymbol{T}(n+1) = \begin{bmatrix}
\boldsymbol{I} & \boldsymbol{0} & . & . & \boldsymbol{0} & \boldsymbol{0} \\
\boldsymbol{CK} & \boldsymbol{I} & . & . & \boldsymbol{0} & \boldsymbol{0} \\
. & . & . & . & . & . \\
. & . & . & . & . & . \\
\boldsymbol{CA}^{n-2}\boldsymbol{K} & \boldsymbol{CA}^{n-3}\boldsymbol{K} & . & . & \boldsymbol{I} & \boldsymbol{0} \\
\boldsymbol{CA}^{n-1}\boldsymbol{K} & \boldsymbol{CA}^{n-2}\boldsymbol{K} & . & . & \boldsymbol{CK} & \boldsymbol{I}
\end{bmatrix}
\tag{4.63}
$$

Due to the special structure of this matrix, it will always be positive definite. This implies that the moving average order only in special cases will be n-1. In these special cases the following relation must be fulfilled

$$
\boldsymbol{C}_n = \boldsymbol{A}_n + \boldsymbol{A}_{n-1}\boldsymbol{CK} + \ldots + \boldsymbol{A}_1\boldsymbol{CA}^{n-2}\boldsymbol{K} + \boldsymbol{CA}^{n-1}\boldsymbol{K} = \boldsymbol{0} \tag{4.64}
$$

since this is the only way the order of the moving average polynomial obtained from the innovation state space system can be reduced to n-1. According to theorem 2.1, the steady-state Kalman gain and the innovation covariance matrix of (4.60) are given by

$$\begin{aligned} \boldsymbol{K} &= \left(\boldsymbol{APC}^T + \boldsymbol{B}\Delta\boldsymbol{D}^T + \boldsymbol{S}\right)\Lambda^{-1} = \left(\boldsymbol{APC}^T + \boldsymbol{S}\right)\Lambda^{-1} \\ \Lambda &= \boldsymbol{CPC}^T + \boldsymbol{D}\Delta\boldsymbol{D}^T + \boldsymbol{R} = \boldsymbol{CPC}^T + \boldsymbol{R} \end{aligned} \tag{4.65}$$

Inserting these definitions into (4.64) yields

$$\begin{aligned} \boldsymbol{C}_n\Lambda = {} & \left(\boldsymbol{A}_n\boldsymbol{C} + \boldsymbol{A}_{n-1}\boldsymbol{CA} + \ldots + \boldsymbol{A}_1\boldsymbol{CA}^{n-1} + \boldsymbol{CA}^n\right)\boldsymbol{PC}^T + \\ & \boldsymbol{A}_n\boldsymbol{R} + \boldsymbol{A}_{n-1}\boldsymbol{CS} + \ldots + \boldsymbol{A}_1\boldsymbol{CA}^{n-2}\boldsymbol{S} + \boldsymbol{CA}^{n-1}\boldsymbol{S} \end{aligned} \tag{4.66}$$

By using the (2.39) this relation reduces to

$$\boldsymbol{C}_n = \boldsymbol{A}_n\boldsymbol{R}\Lambda^{-1} + \left(\boldsymbol{A}_{n-1}\boldsymbol{C} + \ldots + \boldsymbol{A}_1\boldsymbol{CA}^{n-2} + \boldsymbol{CA}^{n-1}\right)\boldsymbol{S}\Lambda^{-1} \tag{4.67}$$

If the process noise is limited then the covariance matrices $\boldsymbol{Q}$ and $\boldsymbol{S}$ tend to zero. In this case $\boldsymbol{C}_n$ is equal to $\boldsymbol{A}_n\boldsymbol{R}\Lambda^{-1}$. On the other hand, if the measurement noise is limited the covariance matrices $\boldsymbol{R}$ and $\boldsymbol{S}$ tend to zero, and in this case $\boldsymbol{C}_n$ is zero. The reduction of the moving average due to the lack of measurement noise only applied to the moving average coefficient matrix $\boldsymbol{C}_n$. In other words, it is not possible to eliminate other coefficient matrices $\boldsymbol{C}_i$, for $i < n$. The reason is that the relation (2.39), which was used to eliminate the first part of (4.66), cannot be used in other cases.

So in conclusion:

☞ *The presence of measurement noise will in general increase the moving average polynomial order of an nth order system from n-1 to n.*

The important point here is if the response of an nth order linear and time-invariant system excited by a stationary stochastic excitation is sampled and if the samples are of high quality, implying a high signal-to-noise ratio, then the appropriate discrete-time model is likely to be an ARMAV(n,n-1) model. However, if the signal-to-noise ratio is low the appropriate model is more likely to be an ARMAV(n,n). In the following special case the above results are shown for an ARMAV(2,2) model using a different approach.

4.3.1 A Special Case: An Equivalent ARMAV(2,2) Model

In section 4.2.1, the discrete-time covariance equivalent ARMAV(2,1) model of a second-order combined continuous-time system was derived. This section shows how this model appear, when it is subjected to process and measurement noise. Consider the state space realization (4.44) of the covariance equivalent ARMAV(2,1) model, defined by the system matrices

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{A}_2 & -\boldsymbol{A}_1 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{B}_2 - \boldsymbol{A}_1 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \end{bmatrix}, \quad \boldsymbol{D} = \boldsymbol{0} \qquad (4.68)$$

The auto-regressive part will not be affected by the disturbance. The auto-regressive coefficient matrices $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are therefore still determined by (4.33). According to theorem 2.5, the moving average matrices $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ can for this particular state space realization be defined in terms of the Kalman gain defined in (4.65) as

$$\begin{aligned} \begin{bmatrix} \boldsymbol{C}_1 \\ \boldsymbol{C}_2 \end{bmatrix} &= \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{A}_1 & \boldsymbol{I} \end{bmatrix} \boldsymbol{K} + \begin{bmatrix} \boldsymbol{A}_1 \\ \boldsymbol{A}_2 \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{A}_1 & \boldsymbol{I} \end{bmatrix} \left(\boldsymbol{A}\boldsymbol{P}\boldsymbol{C}^T + \boldsymbol{S} \right) \Lambda^{-1} + \begin{bmatrix} \boldsymbol{A}_1 \\ \boldsymbol{A}_2 \end{bmatrix} \end{aligned} \qquad (4.69)$$

If the covariances $\boldsymbol{P}$ and $\boldsymbol{S}$ are partitioned as

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{P}_{11} & \boldsymbol{P}_{12} \\ \boldsymbol{P}_{21} & \boldsymbol{P}_{22} \end{bmatrix}, \quad \boldsymbol{S} = \begin{bmatrix} \boldsymbol{S}_1 \\ \boldsymbol{S}_2 \end{bmatrix} \qquad (4.70)$$

it is possible to express (4.69) as

$$\begin{bmatrix} \boldsymbol{C}_1 \\ \boldsymbol{C}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{A}_1 & \boldsymbol{I} \end{bmatrix} \left(\begin{bmatrix} \boldsymbol{P}_{21} \\ -\boldsymbol{A}_2\boldsymbol{P}_{11} - \boldsymbol{A}_1\boldsymbol{P}_{21} \end{bmatrix} + \begin{bmatrix} \boldsymbol{S}_1 \\ \boldsymbol{S}_2 \end{bmatrix} \right) \Lambda^{-1} + \begin{bmatrix} \boldsymbol{A}_1 \\ \boldsymbol{A}_2 \end{bmatrix} \qquad (4.71)$$

The matrices $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ are then given by the following two equations

$$\boldsymbol{C}_1 = \boldsymbol{A}_1 + \left(\boldsymbol{P}_{21} + \boldsymbol{S}_1 \right) \Lambda^{-1}, \quad \boldsymbol{C}_2 = \boldsymbol{A}_2 \left(\boldsymbol{I} - \boldsymbol{P}_{11} \Lambda^{-1} \right) + \boldsymbol{S}_2 \Lambda^{-1} \qquad (4.72)$$

If the measurement noise is insignificant, then $\boldsymbol{R}$ and $\boldsymbol{S}$ tend to zero. In this case, it follows from (4.65) that the innovation covariance Λ reduces to $\boldsymbol{C}\boldsymbol{P}\boldsymbol{C}^T = \boldsymbol{P}_{11}$, which implies that $\boldsymbol{C}_2 = \boldsymbol{0}$. Thus in situations, where the measurement noise is insignificant, the appropriate model will be an ARMAV(2,1) model no matter how much process noise being present.

4.3.2 Example 4.2: An ARMA(2,2) Model

In section 4.2.2 the noise-free covariance equivalent ARMA(2,1) model was derived on the basis of the second-order continuous-time system. Now assume that a zero-mean Gaussian distributed measurement noise $v(t_k)$ is present. This white noise is fully described by the variance R. A state space realization of the ARMA(2,1) model that includes this noise term is given by, see (4.59)

$$\begin{aligned} \boldsymbol{x}(t_{k+1}) &= \boldsymbol{A}\boldsymbol{x}(t_k) + \boldsymbol{B}u(t_k) \ , \quad u(t_k) \in NID(0,\Delta) \\ y(t_k) &= \boldsymbol{C}\boldsymbol{x}(t_k) + v(t_k) \end{aligned} \tag{4.73}$$

with $\{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}\}$ defined as

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -A_2 & -A_1 \end{bmatrix} , \quad \boldsymbol{B} = \begin{bmatrix} 1 \\ B_2 - A_1 \end{bmatrix} , \quad \boldsymbol{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{4.74}$$

and $u(t_k)$ being a zero-mean Gaussian white noise, fully described by the variance Δ. The steady-state covariance matrix $\boldsymbol{P}$ of the state prediction error is obtained as a positive definite solution of the algebraic Riccati equation

$$\boldsymbol{P} = \boldsymbol{A}\boldsymbol{P}\boldsymbol{A}^T + \boldsymbol{B}\Delta\boldsymbol{B}^T - \boldsymbol{A}\boldsymbol{P}\boldsymbol{C}^T\left(\boldsymbol{C}\boldsymbol{P}\boldsymbol{C}^T + R\right)^{-1}\boldsymbol{C}\boldsymbol{P}\boldsymbol{A}^T \tag{4.75}$$

where $\boldsymbol{P}$ is the only unknown. Different techniques exist for solving this special equation, see e.g. Aoki [11]. On the basis of $\boldsymbol{P}$ the variance Λ of the innovations $\boldsymbol{e}(t_k)$, and the moving average parameters C_1 and C_2, are given by

$$\Lambda = \boldsymbol{C}\boldsymbol{P}\boldsymbol{C}^T + R \ , \quad C_1 = A_1 + \frac{\boldsymbol{N}\boldsymbol{P}\boldsymbol{C}^T}{\Lambda} \ , \quad C_2 = A_2\left(1 - \frac{\boldsymbol{C}\boldsymbol{P}\boldsymbol{C}^T}{\Lambda}\right) \tag{4.76}$$

with $\boldsymbol{N} = [\ 0 \ \ 1\ ]$. By comparing the first and last equations in (4.76), it is easy to see that if no measurement noise is present, i.e. $R = 0$, then C_2 is zero. □

4.4 Summary

This chapter has considered what happens when the combined continuous-time system is sampled. The discretization can be performed in a number of ways. However, in the context of ambient excited structures where only the system response is available, an appropriate model can be obtained by the covariance equivalence technique. This technique requires, that the first- and second-order moments of the response of the combined continuous-time system must be equal to the first- and second-order moments of the response of the discretized model at all discrete time instances. In the noise-free case, the discretized model of an *n*th order combined continuous-time system is an ARMAV(*n*,*n*-1) model. If measurement noise is present, then it is shown that the appropriate model changes to an ARMAV(*n*,*n*) model.