

Subspace Identification of Modal Coordinate Time Series

Christopher Hoen, PhD, Principal Engineer, Vetco Aibel AS,
 Bergerveien 12, P.O.Box 81, N-1375 Billingstad, Norway
 christopher.hoen@vetco.com

ABSTRACT

The paper presents how direct estimation of the modal coordinate time series can be performed using a time domain subspace based system identification method. The method overcomes the traditional limitation on maximum number of modal coordinates to be estimated being less or equal to the applied number of sensors. This is achieved by utilizing the information on system order inherent in the state sequence applied as basis for the subspace identification of the state space system matrices.

NOMENCLATURE

$\mathbf{M}, \mathbf{C}, \mathbf{K}$	mass damping and stiffness matrices subscripts s or f refers to structure properties and force related properties respectively	$\mathbf{A}, \mathbf{B}, \mathbf{D}, \mathbf{E}$	state space system matrices, continuous time versions, subscripts a, v, q relates to acceleration, velocity and displacement
$\mathbf{q}(t), \dot{\mathbf{q}}(t), \ddot{\mathbf{q}}(t)$	vector of generalised time dependent displacements, velocities and accelerations	$\bar{\mathbf{A}}, \bar{\mathbf{B}}, \bar{\mathbf{D}}, \bar{\mathbf{E}}, \mathbf{K}_g$	state space system matrices, discrete time, innovation form versions, including Kalman gain
$\mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}, t)$	total dynamic forcing function,	$\mathbf{x}(t), \dot{\mathbf{x}}(t)$	state vector, state vector time derivative
$\mathbf{f}_s(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}, t)$	stochastic part and	$\mathbf{y}(t), \mathbf{u}(t)$	system output and system input
$\mathbf{f}_d(t), \mathbf{f}_r(t)$	deterministic (measured) part, and residual (noise) part	$\mathbf{e}(k), \mathbf{e}_m(t)$	innovation and measurement noise,
$\mathbf{B}_d, \mathbf{u}(t)$	input (load) influence matrix and deterministic inputs	$\mathbf{v}(t), \mathbf{w}(t)$	state output and process noise
$\mathbf{Y}, \hat{\mathbf{Y}}$	matrix of output vectors, matrix of estimated k-step ahead predicted outputs	$\mathbf{X}, \hat{\mathbf{X}}$	state sequence matrix, true and estimate
\mathbf{U}	matrix of deterministic input vectors	\mathbf{W}_i	weighting matrix
$\mathcal{S}_L^d, \mathcal{S}_L^s$	coefficient matrices for deterministic and stochastic excitation in the extended state space model	\mathcal{O}_L	extended observability matrix
$\mathbf{U}_1, \mathbf{S}_1, \mathbf{V}_1$	matrices of an SVD	\mathbf{G}	state estimation coefficient matrix
\mathbf{R}	invertible scaling matrix	t_0	initial time
t	continuous time variable	m, r	number of measured outputs and inputs
k	discrete time variable		
N, L, J	number of samples, future horizon for identification, past horizon for instrumental variables		

Ψ, Ψ^{-1}	right and left eigenvector matrix	Ψ_j, ξ_j^T	column vectors of right and left eigenvector matrices respectively
Λ	diagonal matrix of complex eigenvalues	λ	complex eigenvalue
ω_j	is the damped circular natural frequency of mode j	ω_{0j}	is the undamped circular natural frequency of mode j
α_j	is the damping factor of mode j	ζ_j	is the damping ratio of mode j
$\eta_j(t_0)$	is the initial complex modal coordinate of mode j corresponding to the eigenvalue pair λ_j, λ_j^* and the initial state conditions $\mathbf{x}(t_0)$	φ_j	is the initial modal phase of mode j i.e. $\varphi_j = \arg(a_j(t_0))$
T	Super script T indicates transpose of a matrix	θ_{jk}	is the phase of component k of right state space eigenvector j
*	Superscript $*$ indicates complex conjugate		
\perp	Super script \perp indicates orthogonal complement of a matrix	$E\{\circ\}$	expectation operator
$i = \sqrt{-1}$		$\hat{\circ}$	\wedge over a parameter indicates estimate

INTRODUCTION

Modal analysis and system identification of vibrating structures have so far mainly dealt with the identification of natural frequencies, damping properties and mode shapes. However, when monitoring structures responding to natural excitation it is in many cases of great value to know the vibration amplitude for each excited mode. The standard straight forward way of achieving this is by decoupling the measured response using either experimental modeshapes or modeshapes obtained from a numerical model, as shown by e.g. Kaasen [1] or Hjelm et al [2]. However, one severe drawback with the methods presented so far is that only as many modal coordinates as there are sensors can be identified simultaneously. The present paper will introduce a method for determination of the modal coordinates where this restriction is lifted. The method which is based on the framework of subspace system identification makes it possible to simultaneously determine as many modal coordinates as there are identified natural frequencies and corresponding mode shapes in the data. The method was first applied for modal decomposition of the measured response of a drilling riser, Hoen and Moe [3]. However, in that paper details on the actual method were not given. In this paper we will give some details of the developed method for modal coordinate estimation.

THE EQUATIONS OF MOTION FOR SYSTEM IDENTIFICATION OF VIBRATING STRUCTURES

The dynamic response of a structural system can generally be modelled by a second order differential equation of dimension $(n \times n)$ as follows:

$$\mathbf{M}_s \ddot{\mathbf{q}}(t) + \mathbf{C}_s \dot{\mathbf{q}}(t) + \mathbf{K}_s \mathbf{q}(t) = \mathbf{f}(\ddot{\mathbf{q}}, \dot{\mathbf{q}}, \mathbf{q}, t) = \mathbf{f}_d(t) + \mathbf{f}_s(\ddot{\mathbf{q}}, \dot{\mathbf{q}}, \mathbf{q}, t) \quad (1)$$

where $\ddot{\mathbf{q}}$, $\dot{\mathbf{q}}$ and \mathbf{q} are vectors of generalized acceleration, velocity and displacement, respectively. $\mathbf{f}(\ddot{\mathbf{q}}, \dot{\mathbf{q}}, \mathbf{q}, t)$ is the forcing function which contains known (i.e. measured and thereby deterministic) excitation, $\mathbf{f}_d(t)$, and unknown (stochastic) excitation, $\mathbf{f}_s(\ddot{\mathbf{q}}, \dot{\mathbf{q}}, \mathbf{q}, t)$. \mathbf{M}_s , \mathbf{C}_s and \mathbf{K}_s are the mass, damping and stiffness matrices of the structure. The stochastic part of the forcing function $\mathbf{f}_s(\ddot{\mathbf{q}}, \dot{\mathbf{q}}, \mathbf{q}, t)$ can be decomposed into a sum of elements being proportional to the acceleration, velocity and displacement respectively and a residual which contain all the other load components, also any non-linear effects:

$$\mathbf{M}_s \ddot{\mathbf{q}}(t) + \mathbf{C}_s \dot{\mathbf{q}}(t) + \mathbf{K}_s \mathbf{q}(t) = \mathbf{f}_d(t) + \mathbf{f}_{\ddot{\mathbf{q}}}(t) + \mathbf{f}_{\dot{\mathbf{q}}}(t) + \mathbf{f}_{\mathbf{q}}(t) + \mathbf{f}_r(t) \quad (2)$$

The first three elements of the right hand side of (2) are transferred to the left-hand side of the equation and expressed in terms of the acceleration, velocity and displacement respectively

$$(\mathbf{M}_s + \mathbf{M}_f(t))\ddot{\mathbf{q}}(t) + (\mathbf{C}_s + \mathbf{C}_f(t))\dot{\mathbf{q}}(t) + (\mathbf{K}_s + \mathbf{K}_f(t))\mathbf{q}(t) = \mathbf{f}_d(t) + \mathbf{f}_r(t) \quad (3)$$

$\mathbf{M}_f(t)$, $\mathbf{C}_f(t)$ and $\mathbf{K}_f(t)$ are mass (inertia), damping and stiffness effects caused by the external loading. In case of a structure submerged in water, the effects are known as hydrodynamic added mass and damping, and hydrostatic stiffness effects. The force related parts of the mass, damping and stiffness matrices are generally not time invariant. Therefore the general equation will contain time varying coefficient matrices. However, it is reasonable to assume that they may be regarded as approximately constant. This is at least reasonable in a time scale related to the time characteristics of the system, i.e. natural periods. Then we obtain the following second-order differential equation

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{B}_d\mathbf{u}(t) + \mathbf{f}_r(t) \quad (4)$$

The mass matrix \mathbf{M} is assumed positive definite. The damping matrix \mathbf{C} may contain both viscous damping terms and gyroscopic terms. Gyroscopic terms may occur for e.g. risers with internal flow, and likewise for towed cables, see e.g. Blevins [4], and of course for rotating shafts etc. Thus, the damping matrix may be non-symmetric. The stiffness matrix \mathbf{K} contains general stiffness properties. Normally the stiffness matrix will be symmetric. However, in certain flow-induced vibration problems, e.g. the classical flutter problem of airfoils, the equation of motion may be formulated to yield a non-symmetric stiffness matrix. \mathbf{B}_d is an input influence matrix characterising the locations and type of deterministic inputs $\mathbf{u}(t)$.

The response of the dynamic system can be measured by e.g. accelerometers, inclinometers, rotation rate sensors, strain gages etc. A matrix *output* equation can thus be written as:

$$\mathbf{y}(t) = \mathbf{D}_a\ddot{\mathbf{q}}(t) + \mathbf{D}_v\dot{\mathbf{q}}(t) + \mathbf{D}_q\mathbf{q}(t) + \mathbf{e}_m(t) \quad (5)$$

where the matrices \mathbf{D}_a , \mathbf{D}_v and \mathbf{D}_q are output influence matrices for acceleration, velocity and displacement respectively. $\mathbf{e}_m(t)$ is white measurement noise. The output influence matrices describe the relationship between the vectors $\ddot{\mathbf{q}}$, $\dot{\mathbf{q}}$, \mathbf{q} and the measurement vector \mathbf{y} . Thus, a measured output may be a combination of e.g. acceleration and rotation. This is in fact the case for accelerations measured with linear accelerometers mounted perpendicular to the longitudinal axis of a deep water riser as applied in offshore oil and gas exploration and production. For motions with a long period, the influence of the acceleration of gravity (the " $g \cdot \sin(\theta)$ " component) may exceed the lateral acceleration in magnitude. This needs special attention during analysis of the measurements.

In the case of interpreting system matrices identified or estimated from measured response, i.e. system identification, the system matrices cannot be assumed symmetric even if the tested system should yield symmetric matrices in theory. One major reason for non-symmetry in the identified matrices is that measurements always are imperfect and noisy. Thus, only under very special circumstances the eigenvalue problem of a system given by (4) will become symmetric and positive definite and thereby have real eigenvectors. In the general case complex eigenvectors occur.

The eigenvalue problem corresponding to (4) can be solved in two ways, either by direct solution of the corresponding quadratic eigenvalue problem or as will be done here, by recasting (4) into a first order system in state space form. The state space model is a robust and good engineering model with a good numerical foundation for treating linear vibrating systems and it is as easy to understand as the second order approach.

A STATE-SPACE MODEL

Identification of the system parameters \mathbf{M} , \mathbf{C} , \mathbf{K} , which in modal form are given by natural frequencies, modal damping ratios and mode shapes are not straightforward. The system identification methods applied in experimental modal analysis today are to a large extent based on a reformulation of the second order model (4) into a first order state-space description. See e.g. Juang [5]. In particular state space formulations have been applied for the purpose of system identification of offshore structures; see e.g. Hansteen [6], Hoen [7], Prevosto et al. [8]. Procedures for transformation of the second order model to state- space form can be found in textbooks on structural dynamics or system identification theory, see e.g. Hurty and Rubinstein [9], Juang [5] or Meirovitch [10]. By such procedures it is easy to see that it is always possible to represent a linear system given by (4) and (5) in state space form as follows:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{w}(t) \\ \mathbf{y}(t) &= \mathbf{D}\mathbf{x}(t) + \mathbf{E}\mathbf{u}(t) + \mathbf{v}(t)\end{aligned}\quad (6)$$

With reference to (4), (5) and (6) the following definitions apply

$$\begin{aligned}\mathbf{x}(t) &= \begin{bmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{bmatrix} && \text{is the state vector} \\ \mathbf{A} &= \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix} && \text{is the state transition matrix} \\ \mathbf{B} &= \begin{bmatrix} \mathbf{0} \\ -\mathbf{M}^{-1}\mathbf{B}_d \end{bmatrix} && \text{is the deterministic input matrix} \\ \mathbf{D} &= [\mathbf{D}_q - \mathbf{D}_a\mathbf{M}^{-1}\mathbf{K}, \quad \mathbf{D}_v - \mathbf{D}_a\mathbf{M}^{-1}\mathbf{C}] && \text{is the output matrix} \\ \mathbf{E} &= \mathbf{D}_a\mathbf{M}^{-1}\mathbf{B}_d && \text{is the deterministic feed-through matrix} \\ \mathbf{v}(t) &= \mathbf{D}_a\mathbf{M}^{-1}\mathbf{B}_d\mathbf{w}(t) + \mathbf{e}_m(t) && \text{is the state output noise} \\ \mathbf{w}(t) &= \begin{bmatrix} \mathbf{0} \\ -\mathbf{M}^{-1}\mathbf{f}_r(t) \end{bmatrix} && \text{is the state process noise}\end{aligned}$$

In case the state process noise $\mathbf{w}(t)$ is non-white, for practical purposes a state-space model can model the noise to yield a residual noise process that is white. This will add noise states to the state vector and corresponding terms to the matrices \mathbf{A} , \mathbf{B} , \mathbf{D} . See e.g. Hoen [7] for details.

Other forms of the state space representation are also possible depending on the definition of the state vector and the properties of the matrices \mathbf{M} , \mathbf{C} and \mathbf{K} of (4). See e.g. Hurty and Rubinstein [9] or Laub and Arnold [11]. However, choice of formulation is only a matter of importance with respect to numerical implementation. They will all be related by simple coordinate transformations.

A frequently applied alternative formulation to (6) in discrete time is the *innovation form*; see e.g. Ljung [12]

$$\begin{aligned}\mathbf{x}(k+1) &= \bar{\mathbf{A}}\mathbf{x}(k) + \bar{\mathbf{B}}\mathbf{u}(k) + \mathbf{K}_g\mathbf{e}(k) \\ \mathbf{y}(k) &= \bar{\mathbf{D}}\mathbf{x}(k) + \bar{\mathbf{E}}\mathbf{u}(k) + \mathbf{e}(k)\end{aligned}\quad (7)$$

where \mathbf{K}_g is the Kalman gain matrix and the innovation is defined as $\mathbf{e}(k) = \mathbf{y}(k) - E\{\mathbf{y}(k) | \mathbf{y}(k-1)\}$ where $E\{\cdot\}$ is the expectation operator. The system matrices $\bar{\mathbf{A}}$, $\bar{\mathbf{B}}$, $\bar{\mathbf{D}}$, $\bar{\mathbf{E}}$ are the discrete time equivalents of the

matrices $\mathbf{A}, \mathbf{B}, \mathbf{D}, \mathbf{E}$ of (6). The innovation formulation is particularly useful for estimating the state vector time series, since it is known to yield optimal estimates of the state vector, see e.g. Maybeck [13].

THE STATE SPACE MODAL FORM

The state space models (6) or (7) can be decoupled into a set of $2n$ uncoupled equations applying the eigenvalue decomposition of the state transition matrix, see Hoen [14, 3] for details.

$$\Psi^{-1} \mathbf{A} \Psi = \Lambda \quad (8)$$

where

Λ is the diagonal matrix of eigenvalues of \mathbf{A}
 Ψ^{-1}, Ψ is the left and right eigenvector matrices of \mathbf{A}

The eigenvector matrix of the state space model can be partitioned as

$$\Psi = \begin{bmatrix} \Psi_q \\ \Psi_q \Lambda \end{bmatrix} \quad (9)$$

where Ψ_q is the components of the eigenvectors corresponding to the generalised displacements. Thus we obtain the following modal state space description by applying (8) to e.g. (6)

$$\begin{aligned} \dot{\boldsymbol{\eta}}(t) &= \Lambda \boldsymbol{\eta}(t) + \Psi^{-1} \mathbf{B} \mathbf{u}(t) + \Psi^{-1} \mathbf{w}(t) \\ \mathbf{y}(t) &= \mathbf{D} \Psi \boldsymbol{\eta}(t) + \mathbf{E} \mathbf{u}(t) + \mathbf{v}(t) \end{aligned} \quad (10)$$

where

$$\boldsymbol{\eta}(t) = \Psi^{-1} \mathbf{x}(t) \quad (11)$$

is the complex vector of *state modal coordinates*.

It is well known that the solutions to (6), (7) and (9) are composed of a homogeneous part associated with the initial conditions, and a steady state solution given by the future deterministic input and process noise. The solution to the homogeneous part is useful for interpretation of resonant vibrations such as e.g. lock-in Vortex Induced Vibrations of deep-water risers. The solution to the free vibration problem associated with (6) or (7) is known to be

$$\mathbf{x}(t) = \Psi e^{\Lambda(t-t_0)} \boldsymbol{\eta}(t_0) \quad (12)$$

where $\boldsymbol{\eta}(t_0)$ is a vector of complex coefficients or initial modal weights

INTERPRETATION OF STATE SPACE MODAL RESPONSE

The magnitude and the phase angle of the complex initial modal coordinate interpret as the initial modal amplitude and the initial modal phase angle. The elements of the initial modal coordinate vector $\boldsymbol{\eta}(t_0)$ can therefore be written as

$$\eta_j(t_0) = \xi_j^T \mathbf{x}(t_0) \quad (13)$$

where ξ_j^T is the column vectors of the left eigenvector matrix Ψ^{-1} . In matrix form the free vibration state response is given

$$\mathbf{x}(t) = \Psi e^{\Lambda(t-t_0)} \Psi^{-1} \mathbf{x}(t_0), \quad t \geq t_0 \quad (14)$$

Assume for simplicity of notation that Λ contains only complex eigenvalues, which then will appear in pairs as (λ_j, λ_j^*) , where the asterisk denote complex conjugate. The free vibration response can then be expressed as the following sum over n components

$$\begin{aligned} \mathbf{x}(t) &= \sum_{j=1}^n \boldsymbol{\Psi}_j e^{\lambda_j(t-t_0)} \boldsymbol{\xi}_j^T \mathbf{x}(t_0) + \boldsymbol{\Psi}_j^* e^{\lambda_j^*(t-t_0)} \boldsymbol{\xi}_j^{*T} \mathbf{x}(t_0) \\ &= \sum_{j=1}^n \boldsymbol{\Psi}_j e^{\lambda_j(t-t_0)} \boldsymbol{\eta}_j(t_0) + \boldsymbol{\Psi}_j^* e^{\lambda_j^*(t-t_0)} \boldsymbol{\eta}_j^*(t_0) \end{aligned} \quad (15)$$

Consider now the polar form of the complex numbers in (15)

$$\begin{aligned} e^{\lambda_j(t-t_0)} &= e^{-\alpha_j(t-t_0)} \cdot e^{-i\omega_j(t-t_0)} \\ \boldsymbol{\Psi}_{jk} &= |\boldsymbol{\Psi}_{jk}| e^{i\theta_{jk}}, & \theta_{jk} &= \arg(\boldsymbol{\phi}_{jk}) \\ \boldsymbol{\eta}_j(t_0) &= |\boldsymbol{\eta}_j(t_0)| e^{i\varphi_j(t_0)}, & \varphi_j(t_0) &= \arg(\boldsymbol{\eta}_j(t_0)) \end{aligned} \quad (16)$$

Substituting for (16) in (15) results in the following expression for element k of the free vibration state response vector

$$x_k(t) = \sum_{j=1}^n 2 |\boldsymbol{\eta}_j(t_0)| |\boldsymbol{\Psi}_{jk}| e^{-\alpha_j(t-t_0)} \cos(\omega_j(t-t_0) + \theta_{jk} + \varphi_j(t_0)) \quad (17)$$

The quantities that appear in (17) interpret as follows:

- ω_j the damped natural frequency of mode j
- $\alpha_j = \omega_j \zeta_j$ the damping coefficient, with ζ_j the modal damping ratio of mode j
- $|\boldsymbol{\Psi}_{jk}|$ the magnitude of component k of right state eigenvector j
- θ_{jk} the phase of component k of right state eigenvector j
- $2 |\boldsymbol{\eta}_j(t_0)|$ the initial modal amplitude of state mode j corresponding to the eigenvalue pair (λ_j, λ_j^*) and the initial condition $\mathbf{x}(t_0)$
- $\varphi_j(t_0)$ the initial modal phase of state mode j corresponding to the eigenvalue pair (λ_j, λ_j^*) and the initial condition $\mathbf{x}(t_0)$

Thus, a generally damped structural system decouples into n real state modes, each with $2n$ components corresponding to generalised displacements and velocities. The modes are defined by means of the complex eigenvectors of the system containing magnitudes and phase angles. The appearance of spatially varying phase angles admits travelling wave behaviour of the mode shape as the oscillation proceeds through a cycle. This is a major and important difference from the spatially synchronous oscillation found for classically damped systems.

We also see that the modal decomposition of a measured response vector time series $\mathbf{y}(t)$ can be obtained from estimates of the state-space system matrices and the corresponding state vector time series. Furthermore, the complex modal coordinates which determine the modal amplitude and the modal phase can be defined by means of the left eigenvectors and the state vector at time t_0 , i.e. the initial condition. This is obtained because of the biorthonormality properties of the complex eigenvectors and the system matrices.

SUBSPACE IDENTIFICATION OF THE STATES

As seen in the previous section a key to estimation of the modal coordinate time series is the availability of the state space vector time series. State space vector time series (or state sequences) are readily obtained through (stochastic) subspace system identification algorithms. Alternatively, an estimate of the state space sequence can be obtained using a Kalman filter approach when the state space system matrices including the Kalman gain matrix are known.

During the last decades considerable effort has been put into construction of algorithms for estimation of the parameters in MIMO (multi-input multi-output) state-space systems. In particular the so-called subspace methods, also known as projection methods, have drawn considerable interest. Over the years several authors have presented methods; for the deterministic case, the stochastic case and also for the combined case with both deterministic and stochastic input. See e.g. Ho and Kalman [15], Kung [16], Hoen [7], Prevosto et al [8], Juang [5], Van Overschee and De Moor [17], Ljung and McKelvey [18], Di Ruscio [19], Ljung [12].

A basic idea behind several subspace methods is to first estimate the state vector time series $\mathbf{x}(t)$, and then by linear least squares procedures estimate the system matrices. An estimate of the state vector time series may be constructed directly from the response measurements or from the corresponding covariance functions by application of standard linear algebra decompositions such as QR and/or SVD. From these decompositions, it is also possible to obtain the system matrices directly without actually computing the state vector time series.

Some of the algorithms presented in the literature are designed for impulse response type data, e.g. the ERA algorithm of NASA, see e.g. Juang [5]. Other algorithms can only handle stochastic systems and other again works for deterministic systems, i.e. systems with measured input. However, the latest developments have led to a unification of the approaches and the construction of algorithms that handle the combined deterministic-stochastic estimation problem and each of them as well. The trend is also towards algorithms that work directly on the data avoiding the sometimes numerically ill conditioned covariance estimation step.

The DSR (Deterministic Stochastic Realization) algorithm of Di Ruscio [19] has some features that are very attractive for application to measured structural response. All the system matrices including an estimate of the state vector time series can be obtained directly from standard linear algebra decompositions (QR and SVD) of a data matrix constructed from the input and output vector time series. The Kalman gain matrix is also computed directly from these decompositions without solving any matrix equations like e.g. nonlinear Ricatti or Lyapunov equations.

The stochastic state space identification methods apply equations of the following type for estimation of the state space vector time series, Di Ruscio [19], Ljung [12]

$$\mathbf{Y} = \mathbf{O}_L \mathbf{X} + \mathbf{S}_L^d \mathbf{U} + \mathbf{S}_L^s \mathbf{V} \quad (18)$$

where

\mathbf{Y} is a matrix of stacked measured output vectors

$$\mathbf{Y} = \begin{bmatrix} \mathbf{y}_k & \mathbf{y}_{k+1} & \mathbf{y}_{k+2} & \cdots & \mathbf{y}_{k+K-1} \\ \mathbf{y}_{k+1} & \mathbf{y}_{k+2} & \mathbf{y}_{k+3} & \cdots & \mathbf{y}_{k+K} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{y}_{k+L-1} & \mathbf{y}_{k+L} & \mathbf{y}_{k+L+1} & \cdots & \mathbf{y}_{k+L+K-2} \end{bmatrix} \quad (19)$$

\mathbf{X} the state sequence matrix defined by

$$\mathbf{X} = [\mathbf{x}_k \quad \mathbf{x}_{k+1} \quad \mathbf{x}_{k+2} \quad \cdots \quad \mathbf{x}_{k+K-1}] \quad (20)$$

\mathbf{U} is a matrix of stacked measured input vectors

$$\mathbf{U} = \begin{bmatrix} \mathbf{u}_k & \mathbf{u}_{k+1} & \mathbf{u}_{k+2} & \cdots & \mathbf{u}_{k+K-1} \\ \mathbf{u}_{k+1} & \mathbf{u}_{k+2} & \mathbf{u}_{k+3} & \cdots & \mathbf{u}_{k+K} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{u}_{k+L} & \mathbf{u}_{k+L+1} & \mathbf{u}_{k+L+2} & \cdots & \mathbf{u}_{k+L+K-1} \end{bmatrix} \quad (21)$$

\mathbf{V} is a matrix of stacked innovations noise vectors is defined by

$$\mathbf{V} = \begin{bmatrix} \mathbf{e}_k & \mathbf{e}_{k+1} & \mathbf{e}_{k+2} & \cdots & \mathbf{e}_{k+K-1} \\ \mathbf{e}_{k+1} & \mathbf{e}_{k+2} & \mathbf{e}_{k+3} & \cdots & \mathbf{e}_{k+K} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{e}_{k+L} & \mathbf{e}_{k+L+1} & \mathbf{e}_{k+L+2} & \cdots & \mathbf{e}_{k+L+K-1} \end{bmatrix} \quad (22)$$

O_L is the *extended observability matrix* for the pair (\mathbf{D}, \mathbf{A})

$$O_L = \begin{bmatrix} \mathbf{D} \\ \mathbf{DA} \\ \vdots \\ \mathbf{DA}^{L-1} \end{bmatrix} \quad (23)$$

and the coefficient matrices for the deterministic and stochastic excitation

$$S_L^d = \begin{bmatrix} \mathbf{E} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{DB} & \mathbf{E} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{DAB} & \mathbf{DB} & \mathbf{E} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{DA}^{L-2}\mathbf{B} & \mathbf{DA}^{L-3}\mathbf{B} & \mathbf{DA}^{L-4}\mathbf{B} & \cdots & \mathbf{E} \end{bmatrix} \mathbf{0}_{Lm \times r} \quad (24)$$

$$S_L^d = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{DK}_g & \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{DAK}_g & \mathbf{DK}_g & \mathbf{I} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{DA}^{L-2}\mathbf{K}_g & \mathbf{DA}^{L-3}\mathbf{K}_g & \mathbf{DA}^{L-4}\mathbf{K}_g & \cdots & \mathbf{I} \end{bmatrix} \mathbf{0}_{Lm \times m} \quad (25)$$

In the definitions above the following subscripts are applied:

k is the discrete time variable

L is the future horizon applied for identification

r is the number of measured inputs

m is the number of measured outputs

$K = N - L - k + 1$ is the number of columns in data matrices. $\mathbf{Y}, \mathbf{U}, \mathbf{V}$

N is the number of samples in the time series

Equation (18) can be applied to obtain estimates of the state sequence. The challenge is to remove the effects of the two input terms \mathbf{U} and \mathbf{V} from the measured output data matrix. The \mathbf{U} term is removed by applying a projection matrix \mathbf{U}^\perp which is the orthogonal projection onto the null-space of the input matrix \mathbf{U} , i.e.

$$\mathbf{U}^\perp = \mathbf{I} - \mathbf{U}^T (\mathbf{U}\mathbf{U}^T)^{-1} \mathbf{U} \quad (26)$$

The \mathbf{V} term is a noise term matrix and is removed by using an approach similar to that of the *instrumental variable method*. The basic idea is to correlate away the noise contribution by using a suitable matrix, see e.g. Di Ruscio [19] or Ljung [12] for details. Choosing the past inputs and outputs as instrumental variables give the following weighting matrix \mathbf{W}_1

$$\mathbf{W}_1 = \begin{bmatrix} \mathbf{y}_0 & \mathbf{y}_1 & \mathbf{y}_2 & \cdots & \mathbf{y}_{K-1} \\ \mathbf{y}_1 & \mathbf{y}_2 & \mathbf{y}_3 & \cdots & \mathbf{y}_K \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{y}_{J-1} & \mathbf{y}_J & \mathbf{y}_{J+1} & \cdots & \mathbf{y}_{J+K-2} \\ \mathbf{u}_0 & \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_{K-1} \\ \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \cdots & \mathbf{u}_K \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{u}_{J-1} & \mathbf{u}_J & \mathbf{u}_{J+1} & \cdots & \mathbf{u}_{J+K-2} \end{bmatrix} \quad (27)$$

J is the past horizon applied for removing the noise components of the outputs. Other possible choices for instrumental variables are past outputs alone or past inputs alone. Then by (18) it can be shown that there is a relation between states and outputs as follows, Di Ruscio [19]:

$$\mathbf{Y}\mathbf{U}^\perp\mathbf{W}_1^T = \mathbf{O}_L\mathbf{X}\mathbf{U}^\perp\mathbf{W}_1^T \quad (28)$$

We will present a way of finding the states based on (18) and (28) according to Ljung [12]. After some manipulations we finally arrive at an equation of the following type for estimating the states

$$\hat{\mathbf{X}} = G\hat{\mathbf{Y}} \quad (29)$$

where

$\hat{\mathbf{X}}$ is the matrix of state vector estimates

$\hat{\mathbf{Y}}$ is the matrix outputs corrected for the influence of the measured inputs and the innovation noises.

$\hat{\mathbf{Y}}$ may be computed as follows, but other possibilities also exist:

$$\hat{\mathbf{Y}} = \mathbf{Y}\mathbf{U}^\perp\mathbf{W}_1^T (\mathbf{W}_1\mathbf{U}^\perp\mathbf{W}_1^T)^{-1} \mathbf{W}_1 \quad (30)$$

The coefficient matrix G is obtained from singular value decomposition (SVD) of the projected and weighted data matrix $\hat{\mathbf{Y}}$

$$\hat{\mathbf{Y}} \approx U_1 S_1 V_1^T \quad (31)$$

as

$$G = R^{-1}U_1^T \quad (32)$$

where the invertible matrix R defines the relation between the left singular matrix U_1 and the observability matrix O_L . Typical choices for the matrix R are either $R = I$, $R = S_1$ or $R = S_1^{1/2}$

Finally one must assure that the correct overall scaling of the states has been obtained, e.g. by application of (7), to obtain correct magnitude of the modal coordinate estimates.

CONCLUSIONS

It has been shown how the time series of the modal coordinates are easily computed using the estimate of the state vector time series computed by a subspace identification approach. The modal coordinate estimation method allows estimation of as many modal coordinates as there are identified modes and overcomes the traditional limitation that the number of modal coordinates that can be estimated are equal to the number of measurements.

REFERENCES

- [1] Kaasen, K.E.: "Identification of Modes of VIV on a Drilling Riser from Measurements of Acceleration and Rate of Rotation", ISOPE 2000, Seattle, USA, pp 23 – 28, May 28 – June 2, 1999
- [2] Hjelm, H.P., Brincker, R., Graugaard-Jensen, J. And Munch, K.: "Determination of Stress Histories in Structures by Natural Input Modal Analysis", paper 165, IMAC XXIII, Jan 31 – Feb3. 2005
- [3] Hoen, C and Moe. G.: "Modal Decomposition of Measured Vortex Induced Response of Drilling Risers", ISOPE 1999, Brest, France, pp 679 – 687, May 30 – June 4, 1999.
- [4] Blevins, R.D.: Flow-Induced Vibration, Second Edition, Kreiger Publishing, Malabar, Florida, USA, 1994
- [5] Juang, J-N.: *Applied System Identification*, Prentice Hall PTR, Englewood Cliffs, New Jersey 07632, USA, ISBN 0-13-079211-X, 1994
- [6] Hansteen O.E.: "Outline of a Multichannel ARMA Method to Estimate Resonance Frequencies and Associated Mode Shapes from Recorded Time Histories of Acceleration", Internal Report 51512-7, Norwegian Geotechnical Institute, 1987
- [7] Hoen, C.: *System Identification of Structures Excited by Stochastic Load Processes*, Dr.ing. thesis 1991:79, Division of Marine Structures, NTNU, 1991
- [8] Prevosto, M et al.: "State Space Formulation: A Solution to Modal Parameter Estimation", *Journal of Sound and Vibration*, 148(2), 329-342, 1991
- [9] Hurty, W.C. and Rubinstein, M.F.: *Dynamics of Structures*, Prentice Hall, Englewood Cliffs, NJ, USA 1964
- [10] Meirovitch, L (1997). *Principles and Techniques of Vibrations*, Prentice Hall International, Upper Saddle River, New Jersey 07458, USA, ISBN 0-13-270430-7, 1997
- [11] Laub, A.J. and Arnold, W.F.: "Controllability and Observability Criteria for Multivariable Linear Second-Order models", IEEE Transactions on Automatic Control, vol AC-29, no 2, pp 163-165, Feb 1984
- [12] Ljung, L.: *System Identification Theory for the User Second Edition*. Prentice Hall System and Information science series, Prentice Hall, Englewood Cliffs, New Jersey. 1999.
- [13] Maybeck, P.: *Stochastic Models, Estimation and Control, Volume 1*. Mathematics in Science and Engineering, vol 141, Academic Press, New York, 1979
- [14] Hoen, C.: "An Engineering Interpretation of the Complex Eigensolution of Linear Dynamic Systems", paper 48, IMAC XXIII, Jan 31 – Feb3. 2005

- [15] Ho, B.L. and Kalman, R.E.: "Effective Construction of Linear state-Variable Models from Input/Output functions", *Regelungstechnik*, vol 14, no 12, pp 545-548, 1968
- [16] Kung, S.Y.: "A New Identification and Model Reduction Algorithm via Singular Value Decomposition", *Proc. 12th Ann. Asilomar Conf. on Circuits, Systems and Computers*, Pacific Grove, CA, Nov 1978, pp 705-714
- [17] Van Overschee, P. and De Moor, B.: *Subspace Identification for Linear Systems*, Kluwer Academic Publishers, ISBN 0-7923-9717-7, 1996
- [18] Ljung, L. and McKelvey, T.: "A Least Squares Interpretation of Sub-Space Methods for System Identification", CDC 1996, Kobe, Japan, 1996
- [19] Di Ruscio, D.: "A Method for Identification of Combined Deterministic Stochastic Systems", in *Application of Computer Aided Time Series Modeling*, M. Aoki and A.M Havenner (Eds.), Lecture Notes in Statistics, Springer-Verlag, ISBN 0-387-94751-5, 1997